Math 308: Week-in-Review 5

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$y' + \frac{2t}{t^2-4}y = \frac{3t^2}{t^2-4}$ in standard form

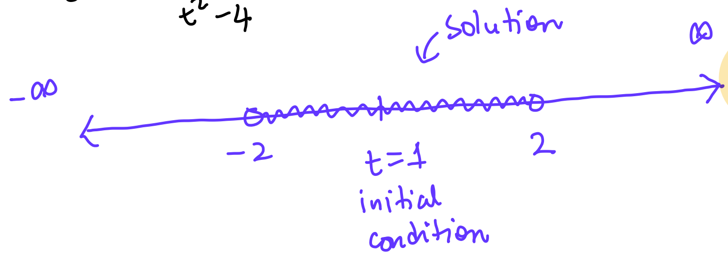
Review for Exam 1

1. For the initial value problem $(t^2-4)y' + 2ty = 3t^2, \qquad y(1) = -3$

   (a) Determine an interval in which the solution to the initial value problem is certain to exist.

   (b) Solve the initial value problem.

(a) a solution of $y' + p(t)y = g(t)$ exists in an interval where $p(t), g(t)$ are continuous, $p(t) = \frac{2t}{t^2-4}$ is continuous if $t \neq \pm 2$ and $g(t) = \frac{3t^2}{t^2-4}$ is continuous at all $t \neq \pm 2$



solution exists inside $(-2,2)$

(b) note that $(t^2-4)y' + 2ty = [(t^2-4)y]'$

* product rule *

* no need to compute integrating factor in this case

$$[(t^2-4)y]' = 3t^2$$

$$(t^2-4)y = 3\int t^2\,dt = \frac{3}{3}t^3 + C$$

$$y = \frac{t^3+C}{t^2-4} \Rightarrow \text{plug in initial condition}$$

$$-3 = \frac{1^3+C}{1^2-4} \Rightarrow 9 = 1 + C \Rightarrow C = 8$$

$$\boxed{y = \frac{t^3+8}{t^2-4}}$$

2. A tank initially contains 10 L of fresh water. Brine containing 20 g/L of salt flows into the tank at a rate of 3 L/min. The solution is kept well stirred and flows out of the tank at a rate of 2 L/min. Determine the concentration of salt in the tank as a function of time.

* Set variables *

$Q(t) \to$ amount of salt at time t in grams (g)

* find differential equation *

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}$$

$$= (\text{concentration}_{\text{in}}) * (\text{flow rate}_{\text{in}}) - (\text{concentration}_{\text{out}}) * (\text{flow rate}_{\text{out}})$$

$$= (20\ g/L) * (3\ L/min) - \left(\frac{Q(t)}{V(t)}\right) * (2\ L/min)$$

$$= 60 - \frac{2Q}{V(t)} \quad \text{where } V(t) = 10 + (3-2)t$$

$V(t) = 10 + t$ * variable volume

$$\begin{cases} \frac{dQ}{dt} = 60 - \frac{2Q}{10+t} \\ Q(0) = 0 \quad \text{(pure water)} \end{cases}$$

* Goal: Find concentration $\frac{Q(t)}{V(t)} = \frac{Q(t)}{10+t}$

* Find $Q(t)$ *

$$Q'(t) + \frac{2}{10+t} Q(t) = 60, \quad p(t) = \frac{2}{10+t}, \quad \mu(t) = e^{\int \frac{2}{10+t} dt} = e^{2\int \frac{1}{10+t} dt} = e^{2\ln(10+t)} = e^{\ln(10+t)^2} = (10+t)^2$$

$$[(10+t)^2 Q(t)]' = 60(10+t)^2$$

$$(10+t)^2 Q(t) = 60\int (10+t)^2 dt = \frac{60}{3}(10+t)^3 + C$$

$$Q(t) = 20(10+t) + C(10+t)^{-2}$$

* Find C *

$$Q(0) = 200 + \frac{C}{100} = 0 \Rightarrow C = -20,000$$

$$Q(t) = 20(10+t) - \frac{20,000}{(10+t)^2}$$

* Concentration *

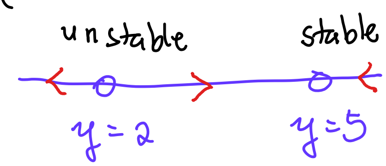
recall $V(t) = 10+t$

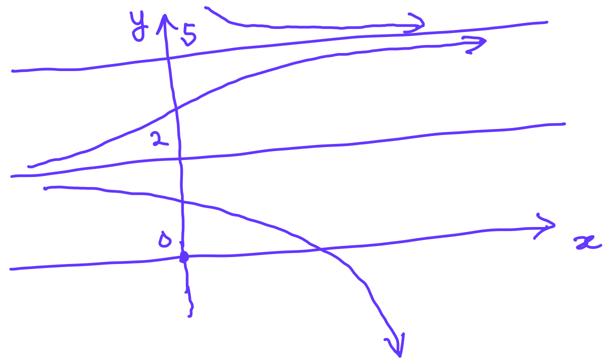
$$\frac{Q(t)}{V(t)} = \frac{20(10+t) - \frac{20,000}{(10+t)^2}}{10+t} = 20 - \frac{20,000}{(10+t)^3}$$

3. Given the differential equation $\dfrac{dy}{dt} = 7y - y^2 - 10$ $= -(y^2 - 7y + 10)$
$= -(y-5)(y-2)$
   (a) Find the equilibrium solutions.
   (b) Sketch the phase line and determine whether the equilibrium solutions are stable, unstable, or semistable.
   (c) Sketch the graph of some solutions.
   (d) Determine the behavior of $y(t)$ as t increases for all possible values of $y(0) = y_0$.
   (e) Solve the equation.

(a) $f(y) = -(y-5)(y-2) = 0 \Rightarrow y = 5,\ y = 2$

(b)





(c) $\lim\limits_{t \to \infty} y(t) = \begin{cases} 5 & \text{if } y_0 \in (2,\infty) \\ 2 & \text{if } y_0 = 2 \\ -\infty & \text{if } y_0 < 2 \end{cases}$

(d) * use separation of variables + partial fractions *

$y' = -(y-5)(y-2) \Rightarrow \displaystyle\int \frac{1}{(y-5)(y-2)}\,dy = \int (-1)\,dx$

$\dfrac{1}{(y-5)(y-2)} = \dfrac{A}{y-5} + \dfrac{B}{y-2} \Rightarrow 1 = A(y-2) + B(y-5) \Rightarrow B = -1/3,\ A = 1/3$

$\displaystyle\int \frac{1}{(y-5)(y-2)}\,dy = \frac{1}{3}\int \left(\frac{1}{y-5} - \frac{1}{y-2}\right)dy = \frac{1}{3}\left[\ln|y-5| - \ln|y-2|\right] = \frac{1}{3}\ln\left|\frac{y-5}{y-2}\right|$

$\ln\left|\dfrac{y-5}{y-2}\right| = \displaystyle\int -3\,dx = -3x + C \Rightarrow \left|\dfrac{y-5}{y-2}\right| = e^{C}e^{-3x} \Rightarrow \dfrac{y-5}{y-2} = Ce^{-3x}$ with $C = \pm e^{C} \neq 0$

* C depends on initial condition*

$y - 5 = yCe^{-3x} - 2Ce^{-3x} \Rightarrow y(1 - Ce^{-3x}) = 5 - 2Ce^{-3x}$

$$\boxed{y = \frac{5 - 2Ce^{-3x}}{1 - Ce^{-3x}}}$$

4. Solve the following equations. If any initial value is given, then solve the initial value problem. If no initial value is given, then find the general solution. Find an explicit solution if possible.

(a) $y' - 2y = x^2 e^{2x}$ * first order linear *

$P(x) = -2$, $\mu(x) = e^{-2x}$ * integrating factor *

$$\underbrace{e^{-2x} y' - 2e^{-2x} y}_{} = x^2 e^{2x} \cdot e^{-2x} = x^2$$

$$(e^{-2x} y)' = x^2 \Rightarrow e^{-2x} y = \int x^2 = \frac{x^3}{3} + C$$

$$\boxed{y = \frac{x^3}{3} e^{2x} + Ce^{2x}}$$

(b) $f'' - 7f' + 12f = 2e^{5t}$, $f(1) = 0$, $f'(1) = -1$ * second order, constant coeff, non-homogeneous * $y = y_c + y_p$

* find homogeneous solution *

$\lambda^2 - 7\lambda + 12 = 0$ * characteristic polynomial *

$(\lambda - 4)(\lambda - 3) = 0 \Rightarrow \lambda = 4, 3 \Rightarrow y_c(t) = c_1 e^{3t} + c_2 e^{4t}$ * general solution of homogeneous eqn *

* find particular solution * Guess $y_p = Ae^{5t}$ ← not homogeneous solution *

$$y_p' = 5Ae^{5t}, \quad y_p'' = 25Ae^{5t}$$

$$y_p'' - 7y_p' + 12y_p = Ae^{5t}(25 - (7)(5) + 12) = 2Ae^{5t} = 2e^{5t} \Rightarrow A = 1$$

$$\Rightarrow y_p = e^{5t}$$

* general solution of nonhomogeneous eqn * $y(t) = c_1 e^{3t} + c_2 e^{4t} + e^{5t}$

* solve for c_1, c_2: $y(1) = c_1 e^3 + c_2 e^4 + e^5 = 0$, $y'(t) = 3c_1 e^{3t} + 4c_2 e^{4t} + 5e^{5t}$

$y'(1) = 3c_1 e^3 + 4c_2 e^4 + 5e^5 = -1$ $\Downarrow$ $c_1 = e^{-3} + e^2$, $c_2 = -e^{-4} - 2e$

$$\boxed{y(x) = (e^{-3} + e^2)e^{3t} - (e^{-4} + 2e)e^{4t} + e^{5t}}$$

(c) $(4t-2y)y' = 4t-4y$ * first order non linear *

$M(t,y) + N(t,y)\,y' = 0$

$\underbrace{(4y-4t)}_{M} + \underbrace{(4t-2y)}_{N}y' = 0 \qquad M_y = 4,\ N_t = 4 \ \checkmark$ exact

$F_t = M = 4y-4t, \quad F_y = N = 4t-2y$

$F(t,y) = \int (4y-4t)\,dt = 4ty - \frac{4t^2}{2} + h(y)$

$F_y = 4t + h'(y) = N = 4t-2y \Rightarrow h'(y) = -2y$

$\Rightarrow h(y) = -y^2 + C$

$$F(t,y) = 4ty - 2t^2 - y^2 + C$$

(d) $y' = ty^2 - t$ * first order non-linear *
separable

Equilibrium solns:
$y^2 - 1 = 0 \Rightarrow y = \pm 1$

$y' = t(y^2-1) \Rightarrow \frac{1}{y^2-1}y' = t$

$\frac{1}{(y-1)(y+1)} = \frac{A}{y-1} + \frac{B}{y+1}$ * partial fractions *

$= A(y+1) + B(y-1)$

$\Rightarrow A = 1/2, \ B = -1/2$

$\int \frac{1}{y^2-1}\,dt = \int \frac{1}{(y-1)(y+1)}\,dt = \int t\,dt$

$\Rightarrow \frac{1}{2}\int \left(\frac{1}{y-1} - \frac{1}{y+1}\right)dt = \int t\,dt$

$\Rightarrow \left[\ln|y-1| - \ln|y+1|\right] = 2\int t\,dt$

$C = \pm e^c \neq 0$

$\Rightarrow \ln\left|\frac{y-1}{y+1}\right| = t^2 + C$

$\Rightarrow \left|\frac{y-1}{y+1}\right| = e^c \cdot e^{t^2} \Rightarrow \frac{y-1}{y+1} = Ce^{t^2} \Rightarrow y(1-Ce^{t^2}) = 1 + Ce^{t^2}$

$$y = \frac{1+Ce^{t^2}}{1-Ce^{t^2}}$$

OR

$$y = -\coth\left(\frac{t^2}{2} + c_1\right)$$

(e) $g'' + 2g' + 2g = 2t, \quad g(0) = 0, \quad g'(0) = 1$

* 2nd order, nonhomogeneous, constant coeffs *

$g = g_c + g_p$

* find homogeneous soln *

$\lambda^2 + 2\lambda + 2 = 0 \Rightarrow \lambda = \dfrac{-2 \pm \sqrt{2^2 - 4\cdot 2}}{2} \Rightarrow \lambda = \dfrac{-2 \pm \sqrt{-4}}{2} = \dfrac{-2 \pm 2i}{2} = -1 \pm i$

$g_c(t) = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t)$

same degree polynomial

* Guess particular soln: $g_p = At + B$

$g_p' = A, \quad g_p'' = 0,$

$$\boxed{\begin{aligned} g_p'' + 2g_p' + 2g_p &= 2A + 2(At + B) \\ &= 2At + (2A + 2B) \\ &= 2t \Rightarrow 2A = 2 \Rightarrow A = 1 \\ 2A + 2B = 0 &\Rightarrow 2B = -2A \\ B &= -1 \\ g_p &= t - 1 \end{aligned}}$$

$g = g_c + g_p = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t) + t - 1$

$g(0) = c_1 - 1 = 0 \Rightarrow c_1 = 1$

$g'(t) = -c_1 e^{-t}\cos(t) - c_1 e^{-t}\sin(t) - c_2 e^{-t}\sin(t) + c_2 e^{-t}\cos(t) + 1$

$g'(0) = -c_1 + c_2 + 1 = c_2 = 1$

$g(t) = e^{-t}\cos t + e^{-t}\sin t + t - 1$

(f) $u'' + 2u' + u = 2e^{-t}$

2nd order nonhomogeneous, const. coeff $\Rightarrow u = u_c + u_p$

* homogeneous solution * $\lambda^2 + 2\lambda + 1 = 0 \Rightarrow (\lambda + 1)^2 = 0 \Rightarrow \lambda = -1$ (repeated root)

$u_c(t) = c_1 e^{-t} + c_2 t e^{-t}$

* find particular solution *

$u_p = Ae^{-t}$ ✗ (homogeneous)

$u_p = Ate^{-t}$ ✗ (homogeneous)

* $u_p = At^2 e^{-t}$ * ✓

$u_p' = 2Ate^{-t} - At^2 e^{-t}$
$= 2Ate^{-t} - u_p$

$u_p'' = 2Ae^{-t} - 2Ate^{-t} - u_p' = 2Ae^{-t} - 4Ate^{-t} + u_p$

$u_p'' + 2u_p' + u_p = 2Ae^{-t} - \cancel{4Ate^{-t}} + \cancel{u_p} + \cancel{4Ate^{-t}} - \cancel{2u_p} + \cancel{u_p} = 2Ae^{-t} = 2e^{-t}$

$\Rightarrow A = 1$

$u_p = t^2 e^{-t}$

$u = c_1 e^{-t} + c_2 t e^{-t} + t^2 e^{-t}$

* general solution of the non-homogeneous equation

5. Solve the differential equation by finding an integrating factor that makes the equation exact

$$e^x y' = e^{3x} + e^x y - e^x$$

$$\underbrace{(e^x - e^x y - e^{3x})}_{M} + \underbrace{e^x}_{N} y' = 0$$

$$M_y = -e^x, \quad N_x = e^x$$

$$\frac{M_y - N_x}{N} = \frac{-2e^x}{e^x} = -2$$

$$\frac{d\mu}{dx} = -2\mu \Rightarrow \mu = e^{-2x}$$

* multiply eqn by μ

$$(e^{-x} - e^{-x} y - e^{x}) + e^{-x} y' = 0$$

$$M_y = -e^{-x}, \quad N_x = -e^{-x} \checkmark \quad \text{exact}$$

$$F_x = e^{-x} - e^{-x} y - e^{x} = M, \quad F_y = e^{-x} = N \Rightarrow F(x,y) = \int e^{-x}\,dy = e^{-x} y + h(x)$$

$$F_x = -y e^{-x} + h'(x) = M$$

$$= -y e^{-x} + \underbrace{e^{-x} - e^{x}}_{h'(x)}$$

$$h(x) = -e^{-x} - e^{x} + C$$

implicit

$$e^{-x} y - e^{-x} - e^{x} = C$$

or

$$y = Ce^{x} + 1 + e^{2x}$$

$\leftarrow$ explicit

6. Verify that $y_1(t) = t$ is a solution of the differential equation

$$t^2 y'' - t(t+2)y' + (t+2)y = 0, \quad (t > 0)$$

then find a second solution y_2 so that $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions.

$$y_1 = t, \; y_1' = 1, \; y_1'' = 0 \Rightarrow t^2 \cdot 0 + t(t+2)\cdot 1 + (t+2)t = 0 \checkmark$$

* y_1 is a soln

* rewrite in standard form *

$$y'' - \frac{t+2}{t} y' + \left(\frac{1}{t} + \frac{2}{t^2}\right) y = 0$$

$$p(t) = -1 - 2/t$$

$$e^{-\int p(t)dt} = e^{\int 1 dt} \cdot e^{\int 2/t \, dt} = e^t \cdot t^2$$

Liouville's formula

$$u = \int \frac{e^{-\int p(t)dt}}{y_1^2}\,dt = \int \frac{t^2 \cdot e^t}{t^2}\,dt = e^t$$

$$y_2 = u y_1 = t e^t$$

* general solution

$$y = c_1 t + c_2 t e^t$$

7. Without solving the initial value problem, determine an interval in which the solution is guaranteed to exist.

$$y' + \ln(t+3)y = \sqrt{16 - t^2}, \quad y(-1) = 3.$$

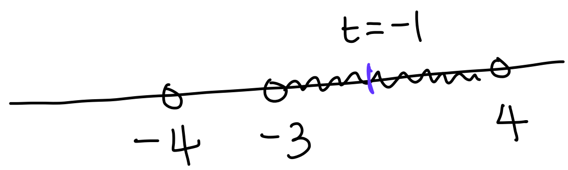
linear $\Rightarrow$ existence & uniqueness when $p(t)$, $g(t)$ continuous

* in standard form* $y' + p(t)y = g(t)$

$t + 3 > 0$ (since $\ln(*)$ defined when $* > 0$)

$t > -3$ and $16 - t^2 \geq 0$ (since $\sqrt{*}$ defined when $* \geq 0$)

$16 \geq t^2 \Leftrightarrow -4 \leq t \leq 4$



solution defined on $(-3, 4)$

8. Without solving the initial value problem, state for which values of t_0 and y_0 the initial value problem is guaranteed to have a unique solution in at least some small interval around t_0.

$$y' = \sqrt{1 - t^2 - y^2}, \quad y(t_0) = y_0$$

non-linear, first order

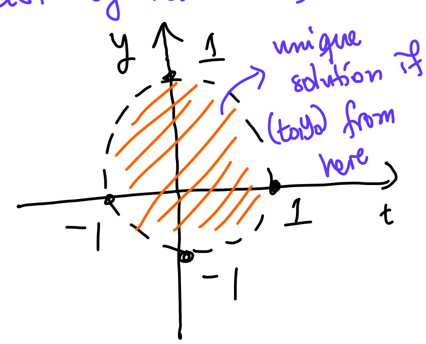
$f(t, y) = \sqrt{1 - t^2 - y^2}$

* Existence & uniqueness if (1) f is continuous near (t_0, y_0)
(2) $\frac{\partial f}{\partial y}$ is continuous near (t_0, y_0)

* f is defined and continuous if $1 - t^2 - y^2 \geq 0$

$1 \geq t^2 + y^2 \Rightarrow t^2 + y^2 \leq 1$ (inside a disk of radius 1)

* $\frac{\partial f}{\partial y} = \frac{-y}{\sqrt{1 - t^2 - y^2}}$ is continuous on $t^2 + y^2 < 1$



9. Solve the initial value problem and determine the interval where the solution exists

$$y' = 3ty^2, \quad y(0) = y_0$$ * non-linear, separable *

$$\int \frac{1}{y^2}\,dy = \int 3t\,dt$$

$$-y^{-1} = \frac{3}{2}t^2 + C$$

$$-\frac{1}{y_0} = C$$

$$y = \frac{-1}{\frac{3}{2}t^2 + C} = \frac{-1}{\frac{3}{2}t^2 - \frac{1}{y_0}}$$

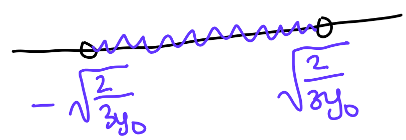
Equilibrium solutions

$y(t) = 0 \ \forall t$ if $y_0 = 0$

$\Rightarrow$ solution exists for $t \in (-\infty, \infty)$

* if $y_0 < 0$ then y exists on $(-\infty, \infty)$

* if $y_0 > 0$, then y exists on $\left(-\sqrt{\frac{2}{3y_0}}, \sqrt{\frac{2}{3y_0}}\right)$



$$\frac{3}{2}t^2 = \frac{1}{y_0} \Rightarrow t^2 = \frac{2}{3}\frac{1}{y_0}$$ end points when $y_0 > 0$

$$\Rightarrow t = \pm\sqrt{\frac{2}{3y_0}}$$