# NOTE ♯4: EXAM 02 REVIEW

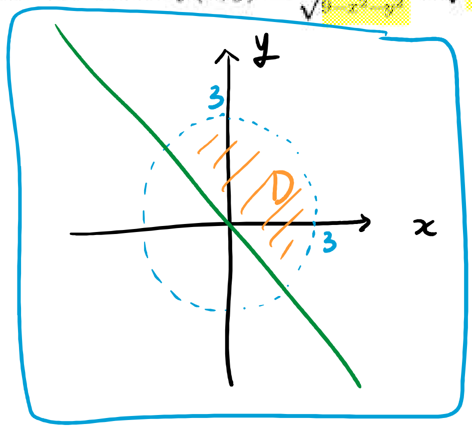
**Problem 1.** Find and sketch the domain of the function $f(x,y) = \frac{1}{\sqrt{9-x^2-y^2}} + \sqrt{x+y}$.

$x+y \geq 0 \Leftrightarrow y \geq -x$

$9-x^2-y^2 > 0 \Leftrightarrow 9 > x^2+y^2$

$$\boxed{D = \{(x,y) \mid y \geq -x,\ 9 > x^2+y^2\}}$$

$y = -x$ $\quad 9 = x^2+y^2$

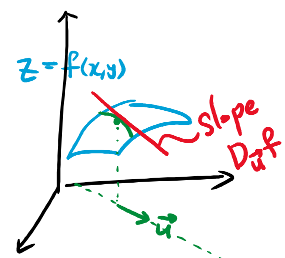



**Problem 2.** Find the directional derivative of $f(x,y,z) = xye^z$ at the point $P(2,4,0)$ in the direction of $Q(3,2,1)$.

$\nabla f = \langle f_x, f_y, f_z \rangle$

$= \langle ye^z, xe^z, xye^z \rangle \big|_{(2,4,0)}$

$= \langle 4, 2, 8 \rangle$




$\vec{u} = \frac{\vec{PQ}}{\|\vec{PQ}\|} = \frac{\langle 3-2, 2-4, 1-0 \rangle}{\sqrt{1^2+(-2)^2+1^2}} = \frac{\langle 1,-2,1 \rangle}{\sqrt{6}}$

$D_{\vec{u}} f = \nabla f \cdot \vec{u}$, if $\|\vec{u}\| = 1$

$D_{\vec{u}} f = \nabla f \cdot \vec{u} = \frac{4 \cdot 1 + 2 \cdot (-2) + 8 \cdot 1}{\sqrt{6}} = \boxed{\frac{8}{\sqrt{6}}}$ rate of change of f

**Problem 3.** Consider the function $f(x,y,z) = x^2 - 2y^2 + 3z^2 + xy$ and a point $P(2,-2,1)$. Find the maximum rate of change of $f$ at the point $P$ and the direction in which it occurs.

$\nabla f = \langle f_x, f_y, f_z \rangle = \langle 2x+y, -4y+x, 6z \rangle \big|_{(2,-2,1)}$

$= \boxed{\langle 2, 10, 6 \rangle}$ direction

$\|\nabla f\| = \sqrt{2^2+10^2+6^2} = \boxed{\sqrt{140}}$ max. rate

$D_{\vec{u}} f = \nabla f \cdot \vec{u} = \|\nabla f\| \, \cancel{\|\vec{u}\|}^{\,1} \cos\theta$

max: $\|\nabla f\|$ when $\theta = 0 \Rightarrow \vec{u} \parallel \nabla f$

min: $-\|\nabla f\|$ when $\theta = \pi \Rightarrow \vec{u} \parallel -\nabla f$

**Problem 4.** Find an equation of the tangent plane to the given surface at the specified point.

$$z = \underbrace{x\sin(x+y)}_{f(x,y)}, \quad (-1, 1, 0)$$

$f(x) \Rightarrow f'(x)$

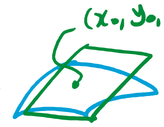
$f(x,y) \Rightarrow f_x$ or $\frac{\partial f}{\partial x}$

$f_x = \sin(x+y) + x\cos(x+y)\big|_{(-1,1,0)}$

$= \sin(0) + (-1)\cos(0) = -1$

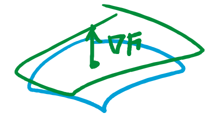
$f_y = x\cos(x+y)\big|_{(-1,1,0)} = -1\cos(0) = -1$

• $z = f(x,y)$ $(x_0, y_0, z_0)$



$\underbrace{z - z_0}_{dz} = f_x(x_0,y_0)\underbrace{(x-x_0)}_{dx} + f_y(x_0,y_0)\underbrace{(y-y_0)}_{dy}$

• $F(x,y,z) = k$



$\nabla F \cdot \langle x-x_0, y-y_0, z-z_0\rangle = 0$

$$\boxed{(F_x)(x-x_0) + (F_y)(y-y_0) + (F_z)(z-z_0) = 0}$$

$z - 0 = (-1)(x+1) + (-1)(y-1)$

$\Leftrightarrow \boxed{z = -(x+1) - (y-1)}$

**Problem 5.** Use the linear approximation(or differentials) of the function $f(x,y) = 1 - xy\cos(\pi y)$ to approximate $f(1.03, 0.98)$.

$f(1.03, 0.98) \approx f(1,1) + f_x(1,1)dx + f_y(1,1)dy$

$f(x,y) \approx f(x_0,y_0) + dz$

$= f(x_0,y_0) + f_x(x_0,y_0)dx + f_y(x_0,y_0)dy$

$f(1,1) = 1 - \cos(\pi) = 1 - (-1) = 2$

$f_x = -y\cos(\pi y)\big|_{y=1} = -\cos(\pi) = 1$

$f_y = -x\left(\cos(\pi y) + y(-\sin(\pi y)(\pi))\right)\big|_{x=y=1} = -\cos(\pi) + \pi\sin(\pi) = 1$

$dx = 1.03 - 1 = 0.03$

$dy = 0.98 - 1 = -0.02$

$(xy\cos(\pi y))_x = \cancel{(x)_x}^{\,1}\, y\cos(\pi y) + x\,\underbrace{\cancel{(y\cos(\pi y))_x}}_{0}$

$f(1.03, 0.98) \approx 2 + 0.03 - 0.02 = \boxed{2.01}$

**Problem 6.** Describe the level curves for

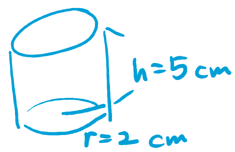
(1) $f(x,y) = \boxed{3x - y = k}$ – level curves are parallel lines.

$y = 3x - k$

(2) $f(x,y) = \sqrt{x^2 + 2y^2} = k$ $\big)\,()^2$ level curves are concentric ellipses

$\boxed{x^2 + 2y^2 = k^2}$

**Problem 7.** The base radius and height of a circular cylinder are measured as $2cm$ and $5cm$, respectively, with possible errors in measurements of as much as $0.1cm$ and $0.2cm$, respectively. Use the differentials to estimate the maximum error in the calculated volume of the cylinder.




$V = f(r,h)$

$dV = f_r\,dr + f_h\,dh$

$$V = \pi r^2 h$$

$$dV = V_r\,dr + V_h\,dh$$

$$= 2\pi r h\cdot dr + \pi r^2\,dh$$

$$= 2\pi\cdot 2\cdot 5\cdot 0.1 + \pi\cdot 2^2\cdot 0.2$$

$$= 2\pi + 0.8\pi = \boxed{2.8\pi\ (cm^3)}$$

**Problem 8.** Find $f_{yx}$ of ~~$f(x,y) = \cos(x^2y)$.~~ $f(x,y) = \cos(x^2y) + y^2e^{3y+\ln(y)+e^{y^2}}$

$f_{yx} = f_{xy}$

$$f_x = -\sin(x^2y)\cdot 2xy + 0 = -2xy\sin(x^2y)$$

$$f_{xy} = -2x\sin(x^2y) + (-2x)y\cos(x^2y)\cdot x^2$$

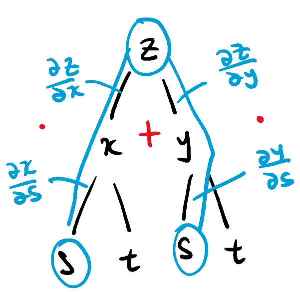
$$= \boxed{-2x\sin(x^2y) - 2x^3y\cos(x^2y)}$$

**Problem 9.** Use the chain rule to calculate $\frac{\partial z}{\partial s}$ when $(s,t) = (1,2)$.

$$z = xe^{xy}, \quad x = st - 1, \quad y = s^2 + t$$

$\Rightarrow$ $x = 1\cdot 2 - 1 = 1$

$y = 1^2 + 2 = 3$




$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial x} = e^{xy} + xye^{xy} = e^3 + 3e^3 = 4e^3$$

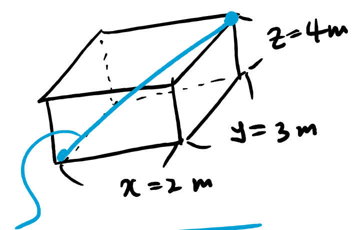
$$\frac{\partial z}{\partial y} = x^2e^{xy} = e^3$$

$$\frac{\partial x}{\partial s} = t\,\Big|_{t=2} = 2$$

$$\frac{\partial y}{\partial s} = 2s\,\Big|_{s=1} = 2$$

$$\frac{\partial z}{\partial s} = 4e^3\cdot 2 + e^3\cdot 2 = \boxed{10e^3}$$

**Problem 10.** The length $x$, width $y$ and height $z$ of a box change with time. At a certain instant, the dimensions are $x = 2m$, $y = 3m$, and $z = 4m$, and $x$, $y$ are increasing at rates $5m/s$, $2m/s$ respectively, while $z$ is decreasing at a rate of $1m/s$. At that same instant, find the rate at which the length of a diagonal is changing.

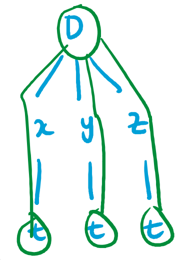



$D = \sqrt{x^2+y^2+z^2}$

$= \sqrt{2^2+3^2+4^2}$

$= \sqrt{4+9+16} = \sqrt{29}$

$$\frac{dD}{dt} = \underset{\frac{2}{\sqrt{29}}}{\frac{\partial D}{\partial x}} \cdot \overset{5}{\frac{dx}{dt}} + \underset{\frac{3}{\sqrt{29}}}{\frac{\partial D}{\partial y}} \cdot \overset{2}{\frac{dy}{dt}} + \underset{\frac{4}{\sqrt{29}}}{\frac{\partial D}{\partial z}} \cdot \overset{-1}{\frac{dz}{dt}}$$




$$\frac{\partial D}{\partial x} = \frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{D} = \frac{2}{\sqrt{29}}$$

$$\frac{\partial D}{\partial y} = \frac{y}{D} = \frac{3}{\sqrt{29}}$$

$$\frac{\partial D}{\partial z} = \frac{z}{D} = \frac{4}{\sqrt{29}}$$

$$\frac{dD}{dt} = \frac{2\cdot5+3\cdot2+4\cdot(-1)}{\sqrt{29}} = \boxed{\frac{12}{\sqrt{29}}\ (m/s)}$$

**Problem 11.** Consider the surface

$$2(x-1)^2 + 3(y-3)^2 - (z-2)^2 = 2, \quad z \geq 0$$

$F(x,y,z) = k$ $\Rightarrow$ $F_x(x-x_0)+F_y(y-y_0)+F_z(z-z_0)=0$

at the point $P$ on the surface determined by $(x, y) = (3, 2)$.

(1) Find the equation of the tangent plane in the form of a linear equation ($ax+by+cz+d = 0$).

$2(3-1)^2+3(2-3)^2-(z-2)^2=2$

$\Leftrightarrow 2\cdot4+3\cdot1-(z-2)^2=2$

$\Leftrightarrow (z-2)^2 = 8+3-2 = 9$

$\Leftrightarrow z-2 = \pm\sqrt{9} = \pm3$

$\Leftrightarrow z = 5, \cancel{-1}$

$\Rightarrow P(3, 2, 5)$

$F_x = 4(x-1)\big|_{x=3} = 8$

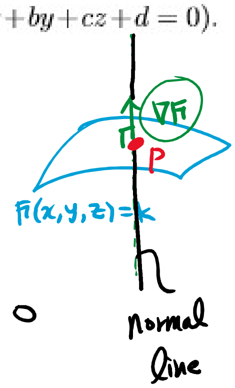
$F_y = 6(y-3)\big|_{y=2} = -6$

$F_z = -2(z-2)\big|_{z=5} = -6$

$8(x-3)+(-6)(y-2)+(-6)(z-5)=0$

$\Leftrightarrow 8x-24-6y+12-6z+30=0$

$\Leftrightarrow \boxed{8x-6y-6z+18=0}$




(2) Find the equation of the normal line in the form of parametric equations.

$x = 3+8t, \quad y = 2-6t, \quad z = 5-6t$

$\nabla f = \vec{0} \Leftrightarrow f_x = 0,\ f_y = 0$

**Problem 12.** Find all critical points and classify them as local maximum, local minimum, or saddle point. You don't have to find the values of $f$.

$$f(x,y) = \frac{2}{3}x^3 + xy^2 + 3x^2 + 2y^2$$

① critical pts:

$$\begin{cases} f_x = 2x^2 + y^2 + 6x = 0 \cdots ⓐ \\ f_y = 2xy + 4y = 0 \cdots ⓑ \Leftrightarrow 2y(x+2) = 0 \Leftrightarrow y = 0 \text{ or } x = -2 \end{cases}$$

- $y=0$: plugging $y=0$ in ⓐ, $2x^2+6x = 2x(x+3) = 0 \Leftrightarrow x = 0, -3$
- $x=-2$: " $x=-2$ in ⓐ, $8+y^2-12=0 \Leftrightarrow y^2 = 4 \Leftrightarrow y = \pm 2$

$\boxed{\text{critical pts: } (0,0),\ (-3,0),\ (-2,2),\ (-2,-2)}$

② Second derivative test:

$$\boxed{\begin{aligned} &D = f_{xx}f_{yy} - f_{xy}^2 \\ &D<0 \Rightarrow \text{saddle} \\ &D>0 \Rightarrow f_{xx}>0 \text{ local min} \\ &\qquad\quad\ f_{xx}<0 \text{ local max} \end{aligned}}$$

$$\left.\begin{aligned} f_{xx} &= 4x+6 \\ f_{xy} &= 2y \\ f_{yy} &= 2x+4 \end{aligned}\right\} \quad D = (4x+6)(2x+4) - (2y)^2$$

- $(0,0)$: $D = 6 \cdot 4 > 0$, $f_{xx} = 6 > 0$

  $\Rightarrow \boxed{\text{local min}}$

- $(-3,0)$: $D = (-6)(-2) - 0 = 12 > 0$

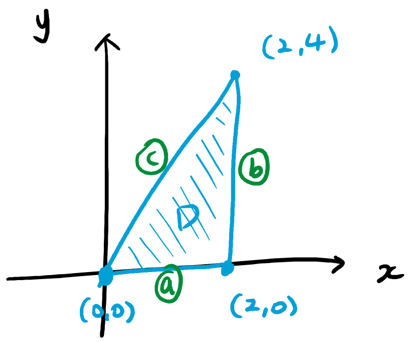
  $f_{xx} = -6 < 0$

  $\Rightarrow \boxed{\text{local max}}$

- $(-2,\pm 2)$: $D = (-2)(0) - 16 = -18 < 0$

  $\Rightarrow \boxed{\text{Saddle pts}}$

**Problem 13.** Let $f(x, y) = x^2 + 2y^2 - 2x - 4y + 1$. Find the absolute maximum and minimum over the triangle with vertices $(0,0)$, $(2,0)$, and $(2,4)$.




① Find all the critical pts in D and evaluate f

② Find the max/min on the boundary of D

③ Compare the results of ① and ② and find the max/min.

① $f_x = 2x - 2 = 0 \Leftrightarrow x = 1$, $f_y = 4y - 4 = 0 \Leftrightarrow y = 1$ $\Rightarrow (1,1) \Rightarrow f(1,1) = 1 + 2 - 2 - 4 + 1 = \boxed{-2}$

② ⓐ : $y = 0,\ 0 \le x \le 2 \rightarrow f(x,0) = x^2 - 2x + 1$ $\quad f'(x,0) = 2x - 2 = 0 \Leftrightarrow x = 1 \Rightarrow f(1,0) = 1 - 2 + 1 = \boxed{0}$

$f(0,0) = \boxed{1}$ $\quad f(2,0) = 2^2 - 2\cdot 2 + 1 = \boxed{1}$

ⓑ : $x = 2,\ 0 \le y \le 4 \Rightarrow f(2,y) = 2^2 + 2y^2 - 2\cdot 2 - 4y + 1 = 2y^2 - 4y + 1$

$f' = 4y - 4 = 0 \Leftrightarrow y = 1$ $\quad f(2,1) = 2\cdot 1 - 4\cdot 1 + 1 = \boxed{-1}$

$f(2,4) = 2\cdot 4^2 - 4\cdot 4 + 1 = 32 - 16 + 1 = \boxed{17}$

ⓒ : $y = 2x,\ 0 \le x \le 2$ $\quad f(x, 2x) = x^2 + 2(2x)^2 - 2x - 4(2x) + 1$

$= x^2 + 8x^2 - 2x - 8x + 1$

$= 9x^2 - 10x + 1$

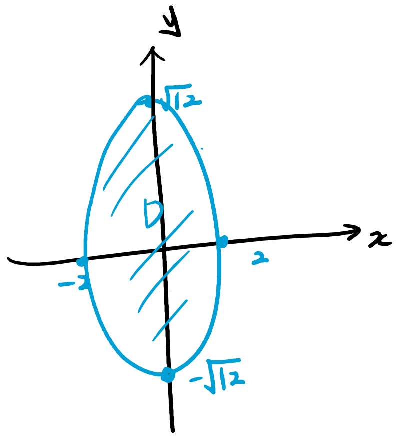
$f' = 18x - 10 = 0 \Leftrightarrow x = \frac{10}{18} = \frac{5}{9}$ $\quad f\big|_{\frac{5}{9}} = 9\cdot\left(\frac{5}{9}\right)^2 - 10\cdot\left(\frac{5}{9}\right) + 1 = \frac{25}{9} - \frac{50}{9} + 1$

$= \boxed{-\frac{16}{9}}$

③ By comparing found values, the max is $\boxed{17}$ the min is $\boxed{-2}$

**Problem 14.** Find the absolute maximum and minimum values of $f$ on the region described by the inequality. Use the Lagrange multiplier method.

$$f(x,y) = 2x^2 + 3y^2, \quad 3x^2 + y^2 \le 12$$




$3x^2 + y^2 = 12$

$y=0: \ 3x^2 = 12 \quad x = \pm 2$

$x=0: \ y^2 = 12 \quad y = \pm\sqrt{12}$

① $\left.\begin{array}{l} f_x = 4x = 0 \\ f_y = 6y = 0 \end{array}\right\} \Leftrightarrow (0,0) \qquad f(0,0) = 2\cdot 0^2 + 3\cdot 0^2 = \boxed{0}$

② Find the max/min on the boundary using the Lagrange multiplier method.

$f(x,y) = 2x^2 + 3y^2 \qquad g(x,y) = 3x^2 + y^2 = 12$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = k \end{cases} \Leftrightarrow \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g = k \end{cases} \Leftrightarrow \begin{cases} 4x = \lambda(6x) & \cdots ⓐ \\ 6y = \lambda(2y) & \cdots ⓑ \\ 3x^2 + y^2 = 12 & \cdots ⓒ \end{cases}$$

From ⓑ $\Leftrightarrow 0 = 2y(\lambda - 3) \Leftrightarrow y = 0$ or $\lambda = 3$

i) $y=0$: Plugging in ⓒ, $3x^2 = 12 \Leftrightarrow x = \pm 2 \quad \therefore f(\pm 2, 0) = 2\cdot(\pm 2)^2 + 0 = \boxed{8}$

ii) $\lambda = 3$: Plugging in ⓐ, $4x = 3(6x) = 18x \Leftrightarrow 0 = 14x \Leftrightarrow x = 0$ Plugging in ⓒ,

$y^2 = 12 \Leftrightarrow y = \pm\sqrt{12} \Rightarrow f(0, \pm\sqrt{12}) = 0 + 3\cdot 12 = \boxed{36}$

③ By comparison, the max is $\boxed{36}$

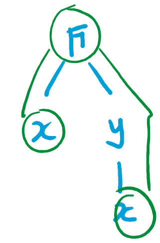
the min is $\boxed{0}$

**Problem 15.** (a) Find $dy/dx$.

$$y\cos(xy) = e^{xy}$$

$$\Leftrightarrow \underbrace{y\cos(xy) - e^{xy}}_{F(x,y)} = 0$$

$$\frac{d}{dx}F(x,y) = \frac{dk}{dx} = 0$$




$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx} = 0$$

$$\boxed{\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{F_x}{F_y}}$$

$$F_x = y(-\sin(xy))y - ye^{xy}$$

$$F_y = \cos(xy) + y(-\sin(xy)\cdot x) - xe^{xy}$$

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y} = \frac{y^2\sin(xy) - ye^{xy}}{\cos(xy) - xy\sin(xy) - xe^{xy}}}$$

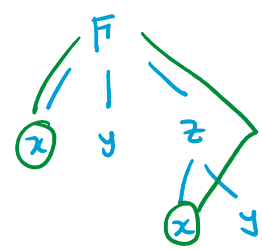
(b) Find $\partial z/\partial x$ and $\partial z/\partial y$.

$$\sin(xyz) = e^{xyz}$$

$$\Leftrightarrow \underbrace{\sin(xyz) - e^{xyz}}_{F(x,y,z)} = 0$$

$$\frac{\partial}{\partial x}F(x,y,z) = \frac{\partial k}{\partial x} = 0$$

$$z = z(x,y)$$




$$F_x + F_z\cdot\frac{\partial z}{\partial x} = 0$$

$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}}$$

$$\boxed{\begin{aligned} F_x &= \cos(xyz)\cdot yz - e^{xyz}\cdot yz \\ F_y &= \cos(xyz)\cdot xz - e^{xyz}\cdot xz \\ F_z &= \cos(xyz)\cdot xy - e^{xyz}\cdot xy \end{aligned}}$$

$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}} \qquad \boxed{\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}}$$