Math 308: Week-in-Review 9

Shelvean Kapita

1. Find the following convolutions using the definition only

* definition *

$$(f*g)(t) = \int_0^t f(t-x)g(x)\,dx$$

(a) $e^t * e^{3t}$ $= \int_0^t e^{t-x}\cdot e^{3x}\,dx$

$$= e^t\int_0^t e^{-x}\cdot e^{3x}\,dx$$

$$= e^t\int_0^t e^{2x}\,dx$$

$$= e^t\cdot\frac{1}{2}\cdot e^{2x}\Big|_0^t$$

$$= e^t\cdot\frac{1}{2}\left(e^{2t}-1\right)$$

$$= \frac{1}{2}e^t\left(e^{2t}-1\right)$$

(b) $t * t^n$, where $n = 0, 1, 2, \cdots$

$$t*t^n = \int_0^t (t-x)x^n\,dx = \int_0^t \left(tx^n - x^{n+1}\right)dx$$

$$= \left[t\frac{x^{n+1}}{n+1}\Big|_0^t - \frac{x^{n+2}}{n+2}\Big|_0^t\right]$$

$$= \frac{t^{n+2}}{n+1} - \frac{t^{n+2}}{n+2}$$

$$= t^{n+2}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{t^{n+2}}{(n+1)(n+2)}$$

$$t*t^n = \frac{t^{n+2}}{(n+1)(n+2)}$$

2. Using the Laplace transform (instead of the definition) compute the following convolutions

(a) $u_a(t) * u_b(t)$

$$u_c(t) = \begin{cases} 1, & t \geq c \\ 0, & 0 \leq t < c \end{cases}$$

$$\mathcal{L}\{u_a(t) * u_b(t)\} = \mathcal{L}\{u_a(t)\}\mathcal{L}\{u_b(t)\}$$

$$= \frac{e^{-as}}{s} \cdot \frac{e^{-bs}}{s}$$

$$= \frac{e^{-(a+b)s}}{s^2}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-(a+b)s}}{s^2}\right\} = u_{a+b}(t)\,(t-a-b)$$

(b) $t^n * t^m$, where $n = 0, 1, 2, \cdots$

$$\mathcal{L}\{t^n * t^m\} = \mathcal{L}\{t^n\}\mathcal{L}\{t^m\}$$

$$= \frac{n!}{s^{n+1}} \cdot \frac{m!}{s^{m+1}}$$

$$= \frac{n!\,m!}{s^{(m+n+2)}}$$

$$\mathcal{L}^{-1}\left\{\frac{n!\,m!}{s^{(m+n+2)}}\right\} = \frac{n!\,m!\,t^{n+m+1}}{(n+m+1)!}$$

$$\mathcal{L}\{t^n * t^m\} = \frac{n!\,m!}{(n+m+1)!}\,t^{n+m+1}$$

3. In each of the following cases find a function (or generalized function) $g(t)$ that satisfies the equality for $t \geq 0$

Let $G(s) = \mathcal{L}\{g(t)\}$

(a) $t * g(t) = t^4$

$\mathcal{L}\{t * g(t)\} = \mathcal{L}\{t^4\}$

$\mathcal{L}\{t\}\mathcal{L}\{g(t)\} = \frac{4!}{s^5} \Rightarrow \frac{1}{s^2} \cdot G(s) = \frac{24}{s^5} \Rightarrow G(s) = 24\frac{s^2}{s^5} = \frac{24}{s^3}$

$G(s) = \frac{24}{s^3} \Rightarrow g(t) = \mathcal{L}^{-1}\{G(s)\} = 12t^2$

$= 12 \cdot \frac{2}{s^3}$

(b) $1 * 1 * g(t) = t^2$

$\mathcal{L}\{1 * 1 * g(t)\} = \mathcal{L}\{1\}\mathcal{L}\{1\}\mathcal{L}\{g(t)\} = \mathcal{L}\{t^2\}$

$\Rightarrow \frac{1}{s} \cdot \frac{1}{s} \cdot G(s) = 2/s^3$

$G(s) = 2\frac{s^2}{s^3} = 2/s$

$g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\{2/s\}$

$g(t) = 2$

(c) $1 * g(t) = 1$

$\mathcal{L}\{1 * g(t)\} = \mathcal{L}\{1\}$

$\cancel{\mathcal{L}\{1\}} \cdot \mathcal{L}\{g(t)\} = \cancel{\mathcal{L}\{1\}}$

$\mathcal{L}\{g(t)\} = 1$

$g(t) = \delta(t)$

4. Write the inverse Laplace transform in terms of a convolution integral

$$F(s) = \frac{s}{(s+1)^2(s+4)^3}$$

$$F(s) = \frac{s}{(s+1)^2} \cdot \frac{1}{(s+4)^3}$$

$$= \frac{s+1-1}{(s+1)^2} \cdot \frac{1}{(s+4)^3}$$

$$= \left[\frac{1}{s+1} - \frac{1}{(s+1)^2}\right] \cdot \frac{1}{(s+4)^3}$$

$* \; F(s) = \mathcal{L}\{f(t)\}$

$* \; \mathcal{L}\{e^{-t}\} = \frac{1}{s+1}$

$* \; \mathcal{L}\{te^{-t}\} = \frac{1}{(s+1)^2}$

$* \; \mathcal{L}\{t^2e^{-4t}\} = \frac{2}{(s+4)^3}$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = (e^{-t} - te^{-t}) * \frac{1}{2}t^2e^{-4t}$$

$$= \frac{1}{2}e^{-t}(1-t) * t^2e^{-4t}$$

$$f(t) = \frac{1}{2}\int_0^t e^{-(t-x)}(1-(t-x)) \cdot x^2 e^{-4x}\,dx$$

or

$$f(t) = \frac{1}{2}\int_0^t e^{-t}(1-t) \cdot (t-x)^2 e^{-4(t-x)}\,dx$$

5. Solve the initial value problem

$$y'' - 2y' - 3y = g(t), \ \ y(0) = 1, \ \ y'(0) = -3.$$

$\mathcal{L}\{g(t)\} = G(s)$

$s^2\mathcal{L}\{y\} - sy(0) - y'(0) - 2s\mathcal{L}\{y\} - 2y(0) - 3\mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$

$\mathcal{L}\{y\}(s^2 - 2s - 3) = G(s) + s - 5$

$\mathcal{L}\{y\} = \frac{G(s)}{s^2-2s-3} + \frac{s-5}{s^2-2s-3}$

$= \frac{G(s)}{(s-3)(s+1)} + \frac{s-5}{(s-3)(s+1)}$

$= \frac{G(s)}{(s-3)(s+1)} + \frac{s-3-2}{(s-3)(s+1)}$

$= G(s)\left[\frac{1}{4(s-3)} - \frac{1}{4(s+1)}\right] + \frac{1}{s+1} - \frac{1}{2(s-3)} + \frac{1}{2(s+1)}$

$= G(s)\left[\frac{1}{4(s-3)} - \frac{1}{4(s+1)}\right] + \frac{3}{2}\frac{1}{s+1} - \frac{1}{2(s-3)}$

$\frac{1}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}$

$1 = A(s+1) + B(s-3)$

$\Rightarrow B = -1/4, \ A = 1/4$

$y(t) = \underbrace{\frac{1}{4}g(t) * \left(e^{3t} - e^{-t}\right)}_{\text{particular solution}} + \underbrace{\frac{3}{2}e^{-t} - \frac{1}{2}e^{3t}}_{\text{homogeneous solution}}$

$y(t) = \frac{3}{2}e^{-t} - \frac{1}{2}e^{3t} + \frac{1}{4}\int_0^t g(t-x)\left[e^{3x} - e^{-x}\right]dx$

6. Determine the radius of convergence for the power series

(a) $\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$

$$\left| \frac{x^{2n+2}}{(n+1)!} \cdot \frac{n!}{x^{2n}} \right| = \frac{x^2}{(n+1)} \xrightarrow{n\to\infty} 0 \text{ for all } x$$

Hence radius of convergence is ∞

Interval of convergence: $(-\infty, \infty)$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n n^2 (x+2)^n}{3^n}$

$$\left| \frac{(-1)^{n+1}(n+1)^2(x+2)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{(-1)^n \cdot n^2 \cdot (x+2)^n} \right|$$

$$= \frac{|x+2|}{3} \frac{(n+1)^2}{n^2} \xrightarrow{n\to\infty} \frac{1}{3}|x+2| < 1$$

$|x+2| < 3$ radius of convergence is 3

$-3 < x+2 < 3 \Rightarrow -5 < x < 1$

Check endpoints: $x=-5$ $\quad \sum_{n=1}^{\infty} \frac{(-1)^n n^2 (-3)^n}{3^n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n} n^2 3^n}{3^n} = \sum_{n=1}^{\infty} n^2 = \infty$ (diverges)

$x=1$ $\quad \sum_{n=1}^{\infty} \frac{(-1)^n n^2 3^n}{3^n} = \sum_{n=1}^{\infty} (-1)^n n^2 \Rightarrow$ diverges

interval of convergence $-5 < x < 1$

7. For the equation $(x^2+1)y'' + xy' - y = 0$

(a) Determine a lower bound for the radius of convergence for the series solutions for the differential equation about $x_0 = 0$.

(b) Seek its power series solution about $x_0 = 0$. Find the recurrence relation.

(c) Find the general term of each solution $y_1(x)$ and $y_2(x)$

(d) Find the first four terms in each of the solutions. Show that $W[y_1, y_2](0) \neq 0$.

(a) singular points when $x = \pm i$. The distance from $x_0 = 0$ to $x = \pm i$ is 1. So a lower bound on the radius of convergence is $|x| < 1$.

(b) $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$

$$(x^2+1)y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^n + \underbrace{\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}}_{\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n}$$

$$xy' = \sum_{n=0}^{\infty} n a_n x^n$$

$$(x^2+1)y'' + xy' - y = \sum_{n=0}^{\infty}\left[n(n-1)a_n + (n+2)(n+1)a_{n+2} + na_n - a_n\right]x^n$$

$$\Rightarrow \sum_{n=0}^{\infty}\left[(n+1)(n-1)a_n + (n+2)(n+1)a_{n+2}\right]x^n = 0$$

recurrence relation $a_{n+2} = -\dfrac{\cancel{(n+1)}(n-1)}{(n+2)\cancel{(n+1)}} a_n$

$$a_{n+2} = -\frac{n-1}{n+2} a_n \qquad n \geq 0$$

$$a_{n+2} = -\frac{n-1}{n+2} a_n \quad , \quad n \geq 0$$

(I) $a_0 = 1, \quad a_1 = 0$

$$a_2 = -\frac{-1}{2} a_0 = \frac{1}{2} a_0 = \frac{1}{2} \qquad (n=0)$$

$$a_3 = -\frac{0}{3} a_1 = 0 \;, \quad a_5 = 0 , \; \dots \quad a_{2n+1} = 0 \text{ for all } n = 0,1,2,3, \dots$$

$$a_4 = -\frac{1}{4} a_2 = -\frac{1}{2} \cdot \frac{1}{4} \qquad (n=2)$$

$$a_6 = -\frac{3}{6} a_4 = \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{3}{6} \qquad (n=4)$$

$$a_8 = -\frac{5}{8} a_6 = -\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{3}{6} \cdot \frac{5}{8} \qquad (n=6)$$

$\vdots$

$$a_{2n} = \frac{(-1)^{n+1} (3 \cdot 5 \cdot 7 \cdots (2n-3))}{2 \cdot 4 \cdot 6 \cdot 8 \cdots 2n} = \frac{(-1)^{n+1} (2n-3)!!}{(2n)!!} \;, \quad n \geq 2$$

$$y_1(x) = 1 + \frac{x^2}{2} + \sum_{n=2}^{\infty} (-1)^{n+1} x^{2n} \frac{(2n-3)!!}{(2n)!!}$$ where $(2n-3)!! = 1 \cdot 3 \cdot 5 \cdots (2n-3)$, $(2n)!! = 2 \cdot 4 \cdot 6 \cdots (2n)$

(II) $a_0 = 0 , a_1 = 1$

$a_0, a_2, a_4, \dots = 0$ for all even indices

$$a_3 = 0 , \quad a_5 = -\frac{2}{5} a_3 = 0 , \quad a_7 = -\frac{4}{7} a_5 = 0 , \dots$$

$$y_2(x) = x$$

(c) $y_1(x) = 1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{48} - \dots \quad , \quad y_2(x) = x$

(d) $W[y_1, y_2] = y_1 y_2' - y_1' y_2 = \left(1 + \frac{x^2}{2} - \frac{x^4}{8} + \dots\right)(1) - \left(\frac{2x}{2} - \frac{4x^3}{8} + \dots\right)(x)$

$$W[y_1, y_2](0) = 1 \neq 0$$