Note #2: Exam 01 Review

Problem 1. (a) What is the radius of the sphere $x^2 + y^2 + z^2 - 2x + 4y - 6z - 2 = 0$?

$\boxed{x^2 + 2ax = (x+a)^2 - a^2}$

$$\Leftrightarrow (x^2-2x) + (y^2+4y) + (z^2-6z) - 2 = 0$$

$$\Leftrightarrow (x-1)^2 - 1 + (y+2)^2 - 4 + (z-3)^2 - 9 - 2 = 0$$

$$\Leftrightarrow \boxed{(x-1)^2 + (y+2)^2 + (z-3)^2 = 16}$$

center : $(1, -2, 3)$

radius : $\sqrt{16} = 4$

(b) What is the intersection of the sphere with the xz-plane?

$y = 0$

$$(x-1)^2 + \overset{=4}{(0+2)^2} + (z-3)^2 = 16$$

$$\Leftrightarrow \boxed{(x-1)^2 + (z-3)^2 = 12}$$

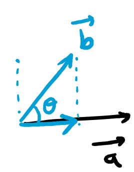
$\Rightarrow$ circle with the center at $(1,0,3)$ and radius $\sqrt{12}$ on the xz plane

Problem 2. Find the scalar and vector projection of $\langle 12, 1, 2\rangle$ onto $\langle -1, 4, 8\rangle$.

$\vec{b} = \langle 12,1,2\rangle$, $\vec{a} = \langle -1,4,8\rangle$

$$\vec{a}\cdot\vec{b} = (12)(-1) + (1)(4) + (2)(8) = -12 + 4 + 16 = 8$$

$$|\vec{a}| = \sqrt{1^2 + 4^2 + 8^2} = \sqrt{1+16+64} = \sqrt{81} = 9$$



$|\vec{a}||\vec{b}|\cos\theta$

$$|\vec{b}|\cos\theta = \boxed{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \mathrm{Comp}_{\vec{a}}\vec{b}}$$

$$\boxed{\mathrm{Proj}_{\vec{a}}\vec{b}} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\frac{\vec{a}}{|\vec{a}|} = \boxed{\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}}$$

- $\boxed{\mathrm{Comp}_{\vec{a}}\vec{b} = \frac{8}{9}}$
- $\boxed{\mathrm{Proj}_{\vec{a}}\vec{b}} = \frac{8}{9^2}\vec{a} = \boxed{\frac{8}{81}\langle -1, 4, 8\rangle}$

$\vec{a}, \vec{b}, \vec{c}$ are three dimensional vectors

Problem 3. Which of the following expressions are meaningful? Select all.
(a) $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ (b) $\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})$ (c) $|\mathbf{a}|(\mathbf{b} \cdot \mathbf{c})$ (d) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$ (e) $(\mathbf{a} \cdot \mathbf{b}) \times (\mathbf{c} \cdot \mathbf{d})$

(a) $\underbrace{3d \cdot 3d}$ → $\underbrace{1d \cdot 3d}$ ✗ not meaningful

(b) $\underbrace{3d \times 1d}$ ✗ not meaningful

(c) $\underbrace{1d \cdot 1d}$ meaningful

(d) $\underbrace{3d \quad 3d}$ meaningful

(e) $\underbrace{1d \times 1d}$ not meaningful.

$|\vec{a}|(\vec{b} \times \vec{c})$ → $(1d)(3d)$ meaningful

Problem 4. Which of the following statements is correct?
(a) $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{i} - \mathbf{j} + \mathbf{k}$ are parallel
(b) $2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $-2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ are orthogonal
(c) None of the above

$$\boxed{\langle a_1, a_2, a_3 \rangle = c\langle b_1, b_2, b_3 \rangle = \langle cb_1, cb_2, cb_3 \rangle \Rightarrow \text{parallel}}$$

$$\boxed{\vec{a} \cdot \vec{b} = 0 \Rightarrow \text{orthogonal (perpendicular)}}$$

(a) $\langle 1, 2, 1 \rangle$, $\times 1$, $\times\left(-\frac{1}{2}\right)$, $\langle 1, -1, 1 \rangle$ ✗ not parallel

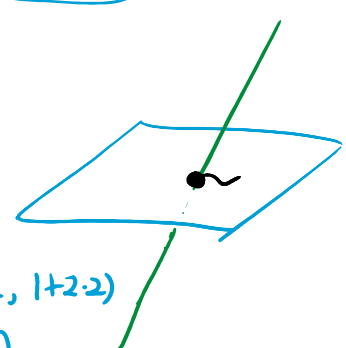
(b) $\langle 2, 2, 1 \rangle \cdot \langle -2, 1, 2 \rangle = (2)(-2) + (2)(1) + (1)(2) = -4 + 2 + 2 = 0$

$\Rightarrow$ orthogonal

Problem 5. Find the point at which the line $x = 2 - t$, $y = 3t$, $z = 1 + 2t$ intersects the plane $2x + 3y - z = 13$.

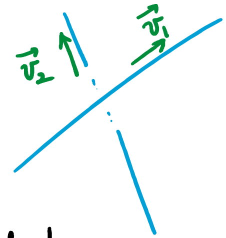
$$2(2-t) + 3(3t) - (1+2t) = 13$$
$$\Leftrightarrow \quad 4 - 2t + 9t - 1 - 2t = 13$$
$$\Leftrightarrow \quad 5t = 10$$
$$\Leftrightarrow \quad t = 2$$

$\boxed{(x, y, z)} = (2-2,\ 3 \cdot 2,\ 1 + 2 \cdot 2) = \boxed{(0, 6, 5)}$



Problem 6. Are these skew lines(do not intersect and are not parallel)?

$$L_1: \quad x = 1 + 2t, \quad y = -2 - t, \quad z = 3 + 4t$$
$$L_2: \quad x = s, \quad y = 2 - s, \quad z = -3 - s$$



(a)

direction vectors $\begin{cases} \vec{v}_1 = \langle 2, -1, 4 \rangle \\ \vec{v}_2 = \langle 1, -1, -1 \rangle \end{cases}$ ×2 ×1 $\Rightarrow$ not parallel!

By (a) & (b), L_1 and L_2 are skew!

(b)

$x: \quad 1 + 2t = s \Leftrightarrow 2t - s = -1 \cdots (1)$

$y: \quad -2 - t = 2 - s \Leftrightarrow -t + s = 4 \cdots (2)$

$z: \quad 3 + 4t = -3 - s \Leftrightarrow 4t + s = -6 \cdots (3)$

(1) + (2): $t = 3$ (2): $s = 7$

(3): $(4)(3) + 7 = 19 \neq -6 \Rightarrow$ no solution!

$\Rightarrow$ no intersecting pt!

Problem 7. a) Find a scalar equation of the plane that passes through the points $P(2, 1, 3)$, $Q(3, -1, 2)$, and $R(4, 2, 4)$.

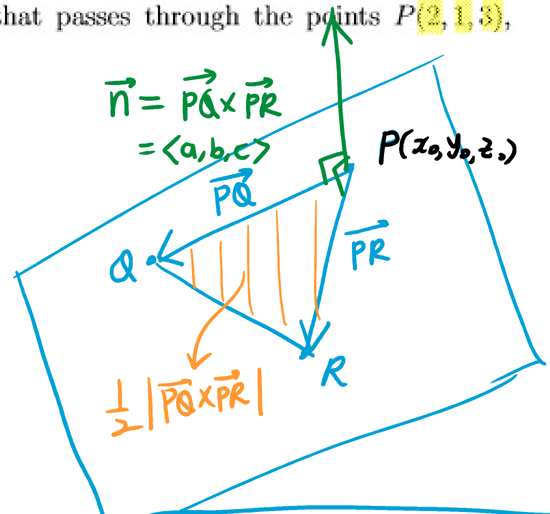
$\vec{PQ} = Q - P = \langle 3-2, -1-1, 2-3 \rangle = \langle 1, -2, -1 \rangle$

$\vec{PR} = R - P = \langle 4-2, 2-1, 4-3 \rangle = \langle 2, 1, 1 \rangle$

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} +\hat{i} & -\hat{j} & +\hat{k} \\ 1 & -2 & -1 \\ 2 & 1 & 1 \end{vmatrix} = ((-2)(1) - (-1)(1))\hat{i} - 3\hat{j} + 5\hat{k}$$

$$= \boxed{-\hat{i} - 3\hat{j} + 5\hat{k}}$$

$$\boxed{-(x-2) - 3(y-1) + 5(z-3) = 0}$$



$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

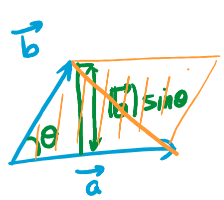
$-x + 2 - 3y + 3 + 5z - 15 = 0$

$\Leftrightarrow \boxed{-x - 3y + 5z - 10 = 0}$

b) Find the area of the triangle determined by P, Q, R.

$|\vec{n}| = \sqrt{1^2 + 3^2 + 5^2} = \sqrt{35}$

$$\boxed{\text{Area} = \frac{\sqrt{35}}{2}}$$

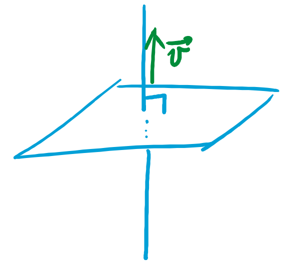


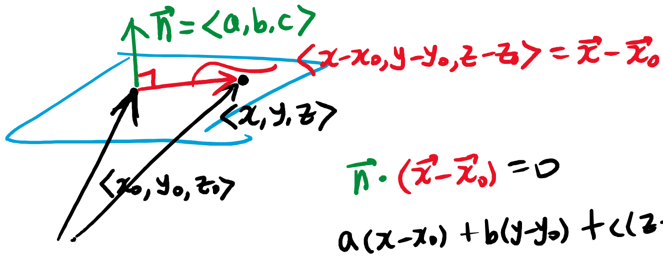
$|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin\theta$

Problem 8. Find the equation of the following planes.

a) The plane passes through the point $(2, 1, -9)$ and is perpendicular to the line $x = 1 - 2t, y = -1 + 3t, z = 5t$.

$$\boxed{-2(x-2)+3(y-1)+5(z+9)=0}$$





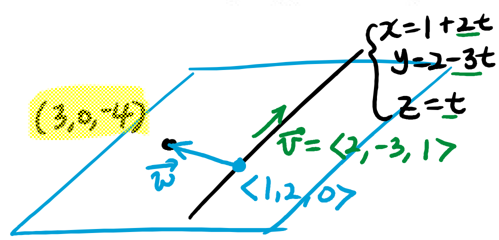
$$\vec{n}\cdot(\vec{x}-\vec{x}_0)=0$$

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

b) The plane passes through the point $(3, 0, -4)$ and contains line $x = 1 + 2t, y = 2 - 3t, z = t$.

$$\vec{w} = \langle 3,0,-4\rangle - \langle 1,2,0\rangle = \langle 2,-2,-4\rangle$$

$$\vec{v}\times\vec{w} = \begin{vmatrix} +\hat{i} & -\hat{j} & +\hat{k} \\ 2 & -3 & 1 \\ 2 & -2 & -4 \end{vmatrix} = 14\hat{i}+10\hat{j}+2\hat{k}$$



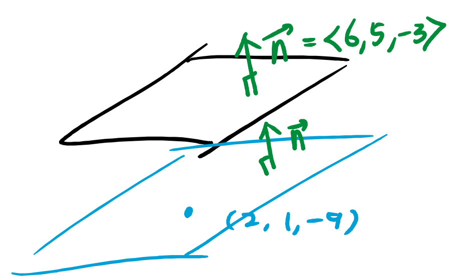
$$\boxed{14(x-3)+10(y-0)+2(z+4)=0}$$

$ax+by+cz+d=0$

c) The plane passes through the point $(2, 1, -9)$ and is parallel to $6x + 5y = 3z + 5$.

$\Leftrightarrow 6x+5y-3z-5=0$

$$\boxed{6(x-2)+5(y-1)-3(z+9)=0}$$



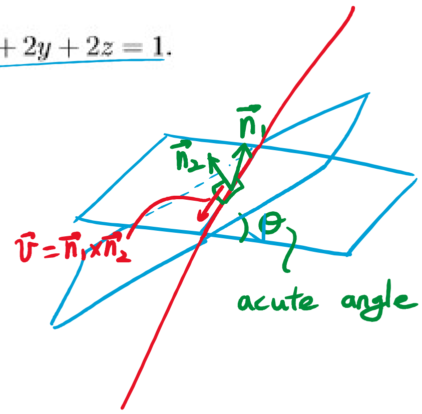
Problem 9. Consider the planes $x + y + z = 2$ and $x + 2y + 2z = 1$.

a) Find the angle between the planes.

$\vec{n}_1 = \langle 1, 1, 1 \rangle$

$\vec{n}_2 = \langle 1, 2, 2 \rangle$

$\vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1||\vec{n}_2| \cos\theta$



$\vec{n}_1 \cdot \vec{n}_2 = 1\cdot1 + 1\cdot2 + 1\cdot2 = 5 > 0$

$|\vec{n}_1| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$

$|\vec{n}_2| = \sqrt{1^2+2^2+2^2} = \sqrt{9} = 3$

$\cos\theta = \dfrac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \dfrac{5}{3\sqrt{3}}$ $\quad \boxed{\theta = \cos^{-1}\left(\dfrac{5}{3\sqrt{3}}\right)}$

b) Find the line of intersection of these two planes.

$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} +\hat{i} & -\hat{j} & +\hat{k} \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix} = \boxed{-\hat{j} + \hat{k}}$

$x = 0$: $\quad y + z = 2$, $\quad 2y + 2z = 1$ $\quad \times$ no solution

$y = 0$: $\quad x + z = 2 \cdots ①$, $\quad x + 2z = 1 \cdots ②$

$② - ①$: $z = -1$

$①$: $x = 3$

$\boxed{(3, 0, -1)}$ $\cdots$ a pt on the line

$\vec{r} = \vec{r}_0 + t\vec{v}$, where $\vec{r} = \langle x, y, z \rangle$, $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$, $\vec{v} = \langle a, b, c \rangle$

$\boxed{\langle x, y, z \rangle = \langle 3, 0, -1 \rangle + t\langle 0, -1, 1 \rangle}$

Vector equation

$\boxed{x = 3, \; y = -t, \; z = -1 + t}$

Parametric eqns

$\boxed{x = 3}$, $\; t = \boxed{\dfrac{y}{-1} = z + 1}$

Symmetric eqn.

Problem 10. Find the domain of the vector function $\mathbf{r}(t) = \langle \frac{t-3}{t-2}, \sin(\sqrt{t+3}), \ln(16 - t^2) \rangle$.

$\dfrac{t-3}{t-2}$: $\quad t - 2 \neq 0 \Leftrightarrow t \neq 2 \cdots ①$

$\sin(\sqrt{t+3})$: $\quad t + 3 \geq 0 \Leftrightarrow t \geq -3 \cdots ②$

$\ln(16 - t^2)$: $\quad 16 - t^2 > 0 \Leftrightarrow 16 > t^2 \Leftrightarrow -4 < t < 4 \cdots ③$

$② + ③$: $-3 \leq t < 4$

$①$: $\Rightarrow -3 \leq t < 2, \; 2 < t < 4$

$\boxed{[-3, 2) \cup (2, 4)}$

Problem 11. Find $\lim_{t\to 1} \mathbf{r}(t)$ where $\mathbf{r}(t) = \left\langle \frac{\sin(\pi t)}{\ln t}, \frac{t-1}{t^2+3t-4}, te^{-2t} \right\rangle$.

$$\lim_{t\to 1} \frac{\sin(\pi t)}{\ln(t)} \overset{\ell}{=} \lim_{t\to 1} \frac{\cos(\pi t)\cdot\pi}{\frac{1}{t}} = \frac{\cos(\pi)\cdot\pi}{\frac{1}{1}} = -\pi$$

$$\lim \frac{f}{g} = \lim \frac{f'}{g'}$$

if $\lim \frac{f}{g} = \frac{\infty}{\infty}$ or $\frac{0}{0}$

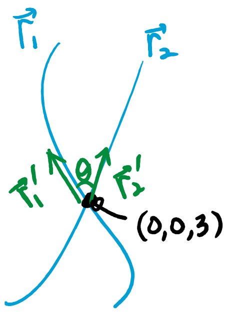
l'Hopital.

$$\lim_{t\to 1} \frac{t-1}{t^2+3t-4} \overset{\ell}{=} \lim_{t\to 1} \frac{1}{2t+3} = \frac{1}{2+3} = \frac{1}{5}$$

$$\lim_{t\to 1} te^{-2t} = e^{-2}$$

$$\Rightarrow \boxed{\langle -\pi, \tfrac{1}{5}, e^{-2} \rangle}$$

Problem 12. Given the curves $r_1(t) = \langle 1 - \cos t, t, 3 - t \rangle$ and $r_2(s) = \langle s^2, \sin(s), 3 + s \rangle$ intersect at the point $(0, 0, 3)$, find the angle of intersection of the two curves.



$\vec{r}_1:\ t=0$ $\qquad$ $\vec{r}_2:\ s^2=0 \Leftrightarrow s=0$

$\vec{r}_1' = \langle \sin t, 1, -1 \rangle$ $\qquad$ $\vec{r}_2' = \langle 2s, \cos(s), 1 \rangle$

$\vec{r}_1'(0) = \langle 0, 1, -1 \rangle$ $\qquad$ $\vec{r}_2'(0) = \langle 0, 1, 1 \rangle$

$$|\vec{r}_1'| = \sqrt{1^2+1^2} = \sqrt{2}$$

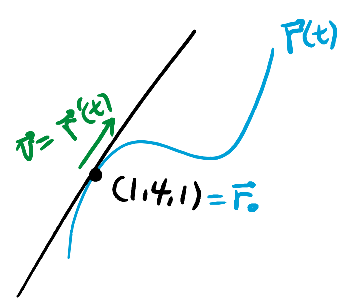
$$|\vec{r}_2'| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\vec{r}_1' \cdot \vec{r}_2' = (0)(0) + (1)(1) + (-1)(1) = 0 \Rightarrow \boxed{\theta = \frac{\pi}{2}}$$

Problem 13. Find parametric equations for the tangent line to the space curve $\mathbf{r}(t) = \left\langle 2t^2 + t + 1, \sqrt{9t+16}, e^{t^2-t} \right\rangle$ at the point $(1, 4, 1)$.

$$\vec{r}'(t) = \left\langle 4t+1, \tfrac{1}{2}(9t+16)^{-\frac{1}{2}}\cdot 9, e^{t^2-t}\cdot(2t-1) \right\rangle$$

$$\sqrt{9t+16} = 4 \Leftrightarrow 9t+\cancel{16} = \cancel{16} \Leftrightarrow t=0$$



$$\vec{r}'(0) = \left\langle 1, \tfrac{9}{8}, -1 \right\rangle$$

$$\boxed{\begin{cases} x = 1 + t \\ y = 4 + \frac{9}{8}t \\ z = 1 - t \end{cases}}$$

Problem 14. Find the unit tangent vector $\mathbf{T}(t)$ to the curve $\mathbf{r}(t) = \langle \sin(2t), -\cos(2t), 4t\rangle$ at the point $(0, 1, 2\pi)$.

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

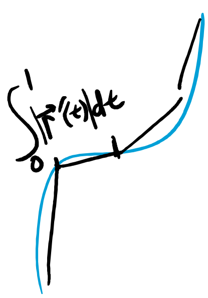
$$\vec{r}\,'(t) = \langle \cos(2t)\cdot 2,\ \sin(2t)\cdot 2,\ 4 \rangle$$

$$4t = 2\pi \iff t = \frac{\pi}{2}$$

$$\vec{r}\,'\left(\tfrac{\pi}{2}\right) = \langle \cos(\pi)\cdot 2,\ \sin(\pi)\cdot 2, 4\rangle = \langle -2, 0, 4\rangle$$

$$\vec{T} = \frac{\langle -2, 0, 4\rangle}{\sqrt{2^2+4^2}} = \frac{1}{\cancel{\sqrt{20}}_{\sqrt{5}}}\langle \cancel{-2}^{-1}, 0, \cancel{4}^{2}\rangle = \left\langle -\frac{1}{\sqrt5}, 0, \frac{2}{\sqrt5}\right\rangle$$

Problem 15. Find the length of the curve $\mathbf{r}(t) = \left\langle 6t, t^2, \frac{1}{9}t^3\right\rangle, 0 \le t \le 1$.



$$\int_0^1 |\vec{r}\,'(t)|\,dt = \int_0^1 6 + \frac{t^2}{3}\,dt$$

$$= \left[6t + \frac{t^3}{9}\right]_0^1$$

$$= 6 + \frac{1}{9} = \boxed{\frac{55}{9}}$$

$$|\vec{r}\,'(t)| = \left|\left\langle 6, 2t, \tfrac{1}{3}t^2\right\rangle\right|$$

$$= \sqrt{6^2 + (2t)^2 + \left(\frac{t^2}{3}\right)^2}$$

$$= \sqrt{\underbrace{36 + 4t^2 + \frac{t^4}{9}}_{\left(6+\frac{t^2}{3}\right)^2}}$$

$$= 6 + \frac{t^2}{3}$$

Problem 16. Find the curvature, κ, of $\mathbf{r}(t) = \langle \cos t, \sin t, 0\rangle$.

$$\kappa = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3}$$

$$\vec{r}\,' = \langle -\sin t, \cos t, 0\rangle$$

$$\vec{r}\,'' = \langle -\cos t, -\sin t, 0\rangle$$

$$\vec{r}\,' \times \vec{r}\,'' = \begin{vmatrix} +\hat{i} & -\hat{j} & +\hat{k} \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \hat{k}$$

$$|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t + 0^2} = \sqrt{1} = 1$$

$$\kappa = \frac{|\hat{k}|}{1^3} = \frac{1}{1^3} = \boxed{1}$$

Problem 17. Given the velocity vector $\mathbf{v}(t) = \langle te^{-t}, \sin(2t), 3t^2 \rangle$ and $\mathbf{r}(0) = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$, find the position vector, $\mathbf{r}(t)$, at time t.

$$\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(s)\,ds$$

$$\int_0^t \underbrace{s}_{f}\underbrace{e^{-s}}_{g'}\,ds = \underbrace{[s(-e^{-s})]_0^t}_{fg} - \underbrace{\int_0^t 1(-e^{-s})\,ds}_{\int f'g} = -te^{-t} + \int_0^t e^{-s}\,ds = -te^{-t} - [e^{-s}]_0^t$$

$$= -te^{-t} - e^{-t} + 1$$

$$\int_0^t \sin(2s)\,ds = \left[-\frac{\cos(2s)}{2}\right]_0^t = -\frac{1}{2}(\cos(2t) - 1)$$

$$\int_0^t 3s^2\,ds = [s^3]_0^t = t^3$$

$$= \langle 2, 1, -1 \rangle + \langle -te^{-t} - e^{-t} + 1, -\tfrac{1}{2}\cos(2t) + \tfrac{1}{2}, t^3 \rangle$$

$$= \boxed{\langle -te^{-t} - e^{-t} + 3, -\tfrac{1}{2}\cos(2t) + \tfrac{3}{2}, t^3 - 1 \rangle}$$