Math 308: Week-in-Review 3

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1. Determine (without solving the problem) an interval in which the solution of the following initial value problem is certain to exist.

   (a)
   $$y' + (\cot x)y = x, \quad y(\pi/2) = 9$$

   (b)
   $$(t^2 - 9)y' + ty = t^4, \quad y(4) = 2$$

   (c)
   $$y' = y\sqrt{t-3}, \quad y(1) = 2$$

2. State where in the ty-plane the hypothesis of the Existence and Uniqueness Theorem are satisfied for the following differential equations

   (a)

$$(2t + 5y)y' = t - y$$

   (b)

$$y' = \frac{\ln(ty)}{y - t^2}$$

3. Solve the following initial value problems and determine how the interval in which the solution exists depends on y_0.

(a) $y' = y^2, \quad y(0) = y_0$

(b) $y' = -\frac{4t}{y}, \quad y(0) = y_0$

4. Verify that both $y_1 = 1 - t$ and $y_2 = -\dfrac{t^2}{4}$ are solutions to the same initial value problem

$$y'(t) = \frac{-t + \sqrt{t^2 + 4y}}{2}, \qquad y(2) = -1.$$

Does the existence of two solutions to the same initial value problem contradict the Existence and Uniqueness Theorem?

5. Given the differential equation

$$y' = y^2(9 - y^2)$$

   (a) Find the equilibrium solutions.
   (b) Graph the phase line. Classify each equilibrium solution as either stable, unstable, or semistable
   (c) Graph some solutions
   (d) If $y(t)$ is the solution of the equation satisfying the initial condition $y(0) = y_0$ for some $y_0 \in (-\infty, \infty)$, find the limit of $y(t)$ as $t \to \infty$

6. Suppose a certain population obeys the logistic equation

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right).$$

If $y_0 = K/4$ find the time τ at which the initial population has doubled. Find the value of τ corresponding to $r = 0.05$.

7. Determine if the differential equation is exact. If it is exact, solve it. You may leave your solution in implicit form.

   (a)
$$(3x^2y + e^y)\, dx + (x^3 + xe^y - 2y)\, dy = 0.$$

   (b)
$$(3x^2y + 8xy^2) + (x^3 + 8x^2y + 12y^2)y' = 0.$$

8. Consider the differential equation

$$(-xy\sin x + 2y\cos x)\,dx + 2x\cos x\,dy = 0.$$

Show that it is not exact, and that it becomes exact when multiplied by the integrating factor $\mu(x, y) = xy$. Solve.