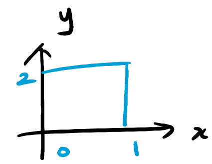
NOTE ♯7: EXAM 03 REVIEW

Problem 1. Calculate the double integral.

$$\iint_R 2xye^{xy^2}\,dA, \quad R = [0,1]\times[0,2]$$



$$\int_0^1\int_0^2 2xye^{xy^2}\,dy\,dx = \int_0^1 \left[e^{xy^2}\right]_{y=0}^{2} dx$$

$u = xy^2$

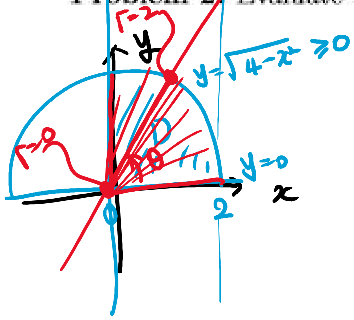
$$= \int_0^1 e^{4x}-1 \;dx$$

$$= \left[\frac{e^{4x}}{4} - x\right]_0^1$$

$$= \left(\frac{e^4}{4}-1\right) - \frac{1}{4}$$

$$= \boxed{\frac{e^4}{4} - \frac{5}{4}}$$

Problem 2. Evaluate the iterated integral.



$$\int_{0}^{2}\int_{0}^{\sqrt{4-x^2}} e^{x^2+y^2}\,dy\,dx$$

$=x$; $\sqrt{4-x^2} = y \iff 4-x^2=y^2 \iff x^2+y^2=4$; $0 = x$; $0 = y$

$x = r\cos\theta$
$y = r\sin\theta$
$x^2+y^2 = r^2$
$dx\,dy = r\,dr\,d\theta$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2} e^{r^2}\, r\,dr\,d\theta$$

$$= \left(\int_0^{\frac{\pi}{2}} d\theta\right)\left[\frac{1}{2}e^{r^2}\right]_{r=0}^{2}$$

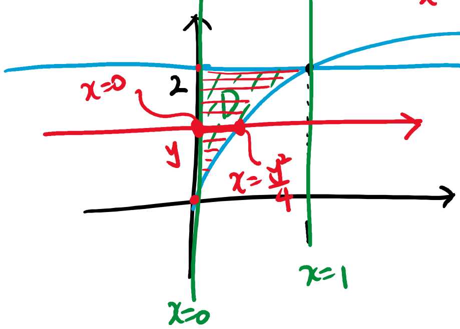
$$= \left(\frac{\pi}{2}\right)\left(\frac{1}{2}(e^4-1)\right) = \boxed{\frac{\pi}{4}(e^4-1)}$$

Problem 3. Sketch the region of integration and evaluate the integral.

$$\int_0^1 \int_{2\sqrt{x}}^2 \sqrt{y^3+1}\,dy\,dx$$

$x = \frac{y^2}{4} \Leftrightarrow y = 2\sqrt{x}$

$y = 2$



$2 = 2\sqrt{x}$

$\sqrt{x} = 1$

$x = 1$

$$= \int_0^2 \int_0^{\frac{y^2}{4}} \sqrt{y^3+1}\,dx\,dy$$

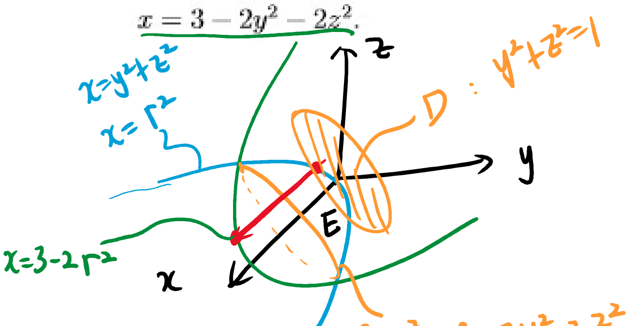
$$= \int_0^2 \frac{y^2}{4}\sqrt{y^3+1}\,dy \qquad u = y^3+1$$

$$= \left[\frac{2}{9}\cdot\frac{1}{4}(y^3+1)^{\frac{3}{2}}\right]_0^2$$

$$= \frac{1}{18}\left((2^3+1)^{\frac{3}{2}} - 1^{\frac{3}{2}}\right)$$

$$= \frac{1}{18}(27-1) = \frac{26}{18} = \boxed{\frac{13}{9}}$$

Problem 4. Find the volume of the solid that is enclosed by the paraboloids $\boxed{x = y^2 + z^2}$ and $x = 3 - 2y^2 - 2z^2$.



$y^2+z^2 = 3-2y^2-2z^2$

$\Leftrightarrow 3(y^2+z^2) = 3$

$\Leftrightarrow y^2+z^2 = 1$

$$V = \iiint_E 1\,dV = \int_0^{2\pi}\int_0^1\left(\int_{r^2}^{3-2r^2} 1\,dx\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(\int_0^1 \left((3-2r^2)-r^2\right) r\,dr\right)d\theta$$

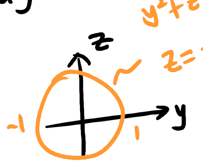
$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^1 3r - 3r^3\,dr\right)$$

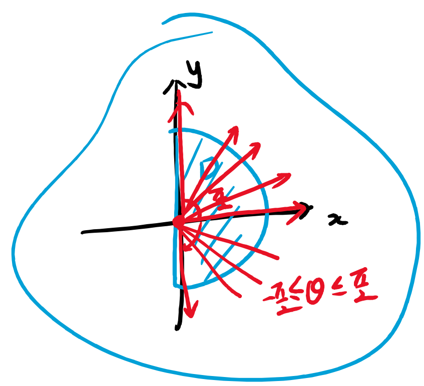
$$= \cdots$$

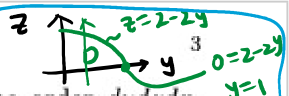
$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{y^2+z^2}^{3-2y^2-2z^2} 1\,dx\,dz\,dy$$

$y^2+z^2=1$

$z = \pm\sqrt{1-y^2}$







$$\int_0^1\int_0^{2-2y}\int_{-\sqrt{y}}^{\sqrt{y}} f\,dx\,dz\,dy$$

$y=x^2$

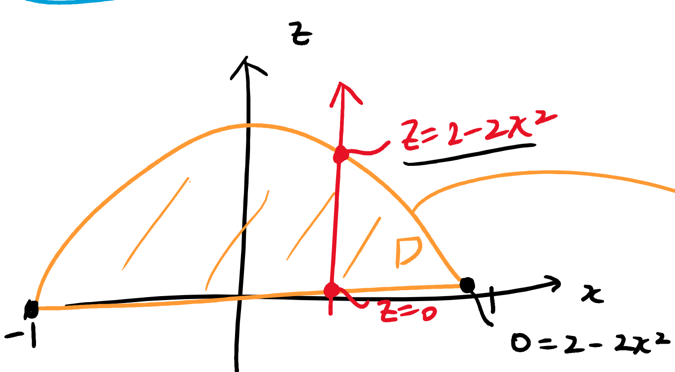
$x=\pm\sqrt{y}$

Problem 5. Express the integral $\iiint_E f(x,y,z)dV$ as an iterated integral in the order $dydzdx$, where E is the solid bounded by the given surfaces.

$$y = x^2, \quad z = 0, \quad 2y + z = 2$$

$z = 2-2y$

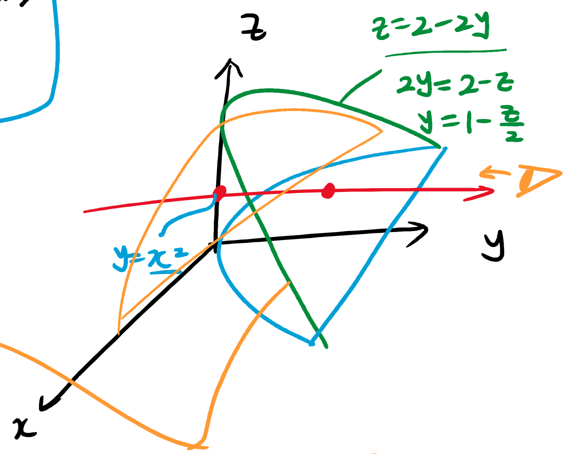
$$\boxed{\int_{-1}^{1}\int_{0}^{2-2x^2}\int_{x^2}^{1-\frac{z}{2}} f(x,y,z)\,dy\,dz\,dx}$$



$0=2-2x^2$

$\Leftrightarrow 2x^2=2$

$\Leftrightarrow x=\pm1$

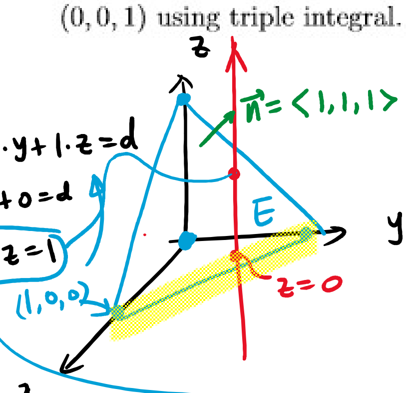


$y=x^2 \rightarrow 2y+z=2$

$\Leftrightarrow 2x^2+z=2$

$\Leftrightarrow z=2-2x^2$

Problem 6. Find the volume of the solid tetrahedron with vertices $(0,0,0), (1,0,0), (0,1,0)$ and $(0,0,1)$ using triple integral.

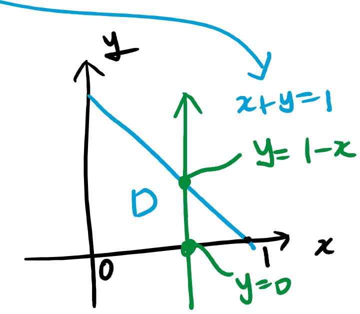


$1\cdot x+1\cdot y+1\cdot z=d$

$1\cdot1+0+0=d$

$x+y+z=1$

$z=1-x-y$



$$V=\iiint_E 1\,dV$$

$$=\boxed{\int_0^1\int_0^{1-x}\int_0^{1-x-y} 1\,dz\,dy\,dx}$$

Problem 7. (a) Change $(2, -2\sqrt{3}, 5)$ from rectangular to cylindrical coordinates.

≈ 1.7

≈ -3.5

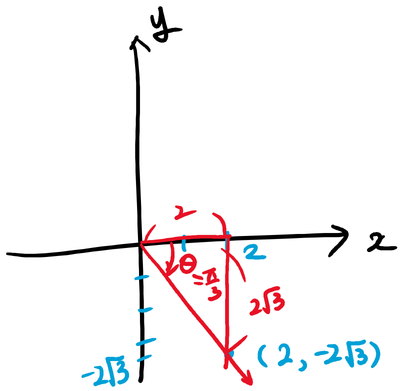
$(x, y, z) \to (r, \theta, z)$

$r = \sqrt{x^2+y^2}$

$x = r\cos\theta$

$y = r\sin\theta$

$\frac{y}{x} = \tan\theta$



$\tan\theta = \sqrt{3}$

$\theta = \frac{\pi}{3}$

$r = \sqrt{2^2+(2\sqrt{3})^2} = \sqrt{4+12} = \sqrt{16} = 4$

$\theta = -\frac{\pi}{3}$

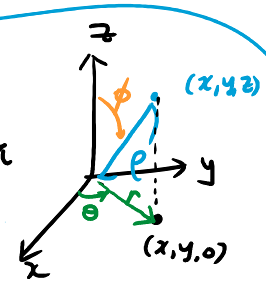
$(r, \theta, z) = (4, -\frac{\pi}{3}, 5)$

(b) Change $(\sqrt{3}, -1, -2\sqrt{3})$ from rectangular to spherical coordinates.

$\boxed{\rho} = \sqrt{\sqrt{3}^2 + 1^2 + (2\sqrt{3})^2} = \sqrt{3+1+12} = \sqrt{16} = \boxed{4}$

$(x, y, z) \to (\rho, \theta, \phi)$

$0 \le \theta \le 2\pi$ $\quad$ $0 \le \phi \le \pi$

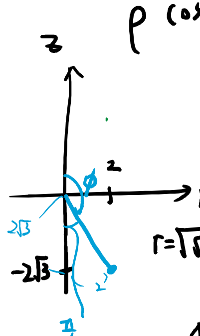


$\rho\cos\phi = z \Leftrightarrow 4\cos\phi = -2\sqrt{3}$

$\Leftrightarrow \cos\phi = -\frac{\sqrt{3}}{2}$

$\Leftrightarrow \phi = \cos^{-1}(-\frac{\sqrt{3}}{2})$

$\boxed{\phi = \frac{5\pi}{6}}$



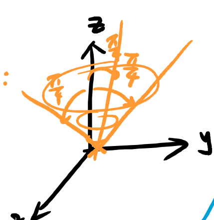
$r = \sqrt{\sqrt{3}^2+1^2} = 2$

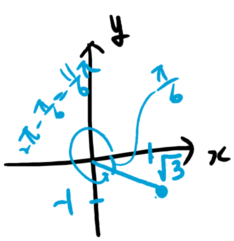
$\rho = \sqrt{x^2+y^2+z^2}$

$\theta \Rightarrow$ polar coordinates of (x,y)

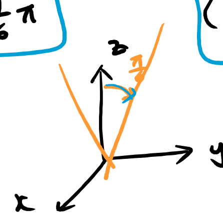
ϕ : $\rho\cos\phi = z$

$\phi = \frac{\pi}{4}$: cone



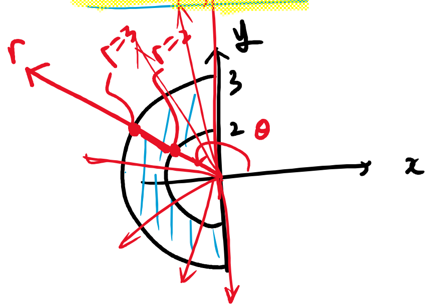


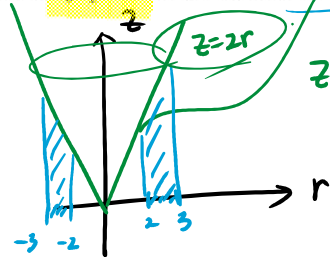
$\boxed{\theta = \frac{11}{6}\pi}$



$\boxed{(\rho, \theta, \phi) = (4, \frac{11}{6}\pi, \frac{5}{6}\pi)}$

Problem 8. Find the volume of the solid below the cone $z = \sqrt{4x^2 + 4y^2}$ and above the ring $4 \le x^2 + y^2 \le 9$, where the ring is in the xy-plane and to the left of the y-axis.





$z = 2\sqrt{x^2+y^2} = 2r$

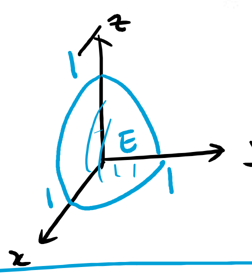
$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_{2}^{3}\int_{0}^{2r} 1 \, dz \; r\,dr\,d\theta = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\left(\int_{2}^{3} 2r^2\,dr\right)d\theta = \left[\frac{2}{3}r^3\right]_2^3\left(\frac{3\pi}{2}-\frac{\pi}{2}\right)$$

$$= \boxed{\frac{2}{3}\left(3^3-2^3\right)\pi}$$

$x^2+y^2=1$

$z \Rightarrow x^2+y^2+z^2=1$

Problem 9. Compute $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} (x^2 + y^2 + z^2)\, dz dy dx$.



$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{1} \rho^2 \, \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

$$= \left(\int_0^1 \rho^4\,d\rho\right)\left(\int_0^{\frac{\pi}{2}} \sin\phi\,d\phi\right)\left(\int_0^{\frac{\pi}{2}} 1\,d\theta\right)$$

$$= \left(\frac{1}{5}\right)\left(\left[-\cos\phi\right]_0^{\frac{\pi}{2}}\right)\left(\frac{\pi}{2}\right)$$

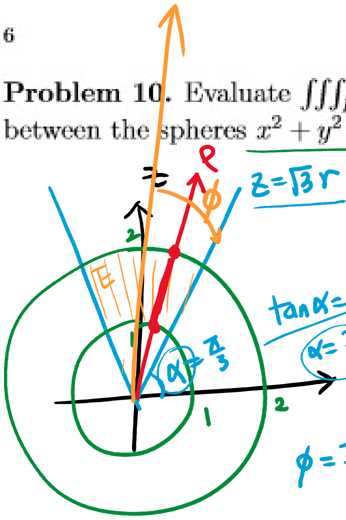
$$= \left(\frac{1}{5}\right)(1)\left(\frac{\pi}{2}\right) = \boxed{\frac{\pi}{10}}$$

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \\ x^2+y^2+z^2 = \rho^2 \end{cases}$$

$+ \; dx\,dy\,dz = \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$

Problem 10. Evaluate $\iiint_E \sqrt{x^2+y^2+z^2}\,dV$, where E lies above the cone $z=\sqrt{3\left(x^2+y^2\right)}$ and between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$.

$z = \sqrt{3}\, r$



$\tan\alpha = \sqrt{3}$

$\alpha = \frac{\pi}{3}$

$\phi = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$

$$= \int_0^{\frac{\pi}{6}}\int_0^{2\pi}\int_1^{2} \rho \;\rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

$$= \left(\int_1^2 \rho^3\, d\rho\right)\left(\int_0^{2\pi} 1\, d\theta\right)\left(\int_0^{\frac{\pi}{6}} \sin\phi\, d\phi\right)$$

$$= \boxed{\cdots}$$

Problem 11. Find the absolute value of the Jacobian of the transformation $x = u^2 + uv$, $y = uv^2$ evaluated at $u = 1, v = -2$.

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2u+v & u \\ v^2 & 2uv \end{vmatrix}$$

$$= (2u+v)(2uv) - uv^2$$

$$= 4u^2v + 2uv^2 - uv^2$$

$$= 4u^2v + uv^2$$

$$= 4(1)^2(-2) + (1)(-2)^2$$

$$= -8 + 4$$

$$= -4$$

$$\iint_R f(x,y)\, dx\,dy = \iint_S f(x(u,v), y(u,v))\, |J|\, du\, dv$$

$$|J| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}\right| = |x_u y_v - x_v y_u|$$

$$|J| = |-4| = \boxed{4}$$

Problem 12. Use the transformation $u = x - y, v = x + y$ to rewrite

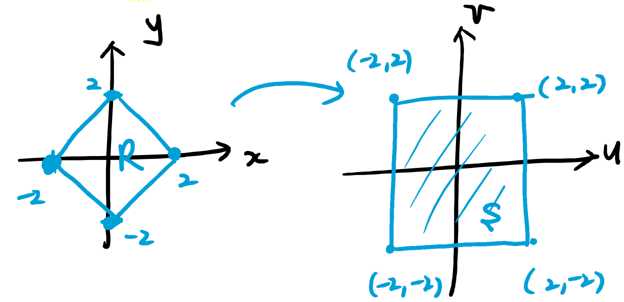
$$\iint_R x^2 - y^2 dA,$$

where R is the square with vertices $(2,0), (0,2), (-2,0)$, and $(0,-2)$. Do not evaluate the integral.

$$\boxed{\iint_R \underbrace{x^2 - y^2}_{\substack{=(x-y)(x+y)\\ = uv}} \underbrace{dA}_{=|J|\,du\,dv} = \int_{-2}^{2}\int_{-2}^{2} uv \; \frac{1}{2}\, du\, dv}$$

$$|J| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}\right| = \left|\frac{1}{2}\right| = \frac{1}{2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1-(-1) = 2$$



$(2,0) \longrightarrow u = 2-0 = 2$, $v = 2+0 = 2$

$(0,2) \longrightarrow u = 0-2 = -2$, $v = 0+2 = 2$

$(-2,0) \longrightarrow \ldots$

$(0,-2) \longrightarrow \ldots$

Problem 13. Use the given transformation to evaluate the integral

$$\iint_R (x + 2y)e^{x^2 - 4y^2} dA$$

where R is the parallelogram enclosed by the lines $\underbrace{x + 2y = 0}_{u=0}, \underbrace{x + 2y = 5}_{u=5}, \underbrace{x - 2y = 0}_{v=0}$, and $\underbrace{x - 2y = 4}_{v=4}$; $u = x + 2y, v = x - 2y$.

$$\iint_R \underbrace{(x+2y)\, e^{x^2-4y^2}}_{u} \underbrace{dA}_{=|J|\,du\,dv}, \quad x^2-4y^2 = \underbrace{(x-2y)}_{v}\underbrace{(x+2y)}_{u}$$

$$= \int_0^5\int_0^4 u\, e^{uv}\, \frac{1}{4}\, dv\, du$$

$$= \int_0^5 \frac{\cancel{u}}{4}\left[\frac{1}{\cancel{u}} e^{uv}\right]_{v=0}^{4} du$$

$$= \int_0^5 \frac{1}{4}(e^{4u} - 1)\, du$$

$$= \left[\frac{1}{16}e^{4u} - \frac{1}{4}u\right]_0^5$$

$$= \left(\frac{1}{16}e^{20} - \frac{1}{4}\cdot 5\right) - \left(\frac{1}{16}e^0 - 0\right)$$

$$= \boxed{\frac{1}{16}e^{20} - \frac{21}{16}}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} = -2-2 = -4$$

$$|J| = \left|\frac{1}{-4}\right| = \frac{1}{4}$$