Math 150 - Week-In-Review 11

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Problem Statements

Exam 3 review

1. Verify the following identities.

a. $\dfrac{3\cot^3 t}{\csc t} = 3\cos t(\csc^2 t - 1)$

$$\frac{3\cot t\cot^2 t}{\csc t} = \frac{3\cdot\frac{\cos t}{\sin t}\cdot\cot^2 t}{\frac{1}{\sin t}} = 3\,\frac{\cos t}{\cancel{\sin t}}\cdot\frac{\cancel{\sin t}}{1}\cdot\cot^2 t = 3\cos t\left(\frac{\cos^2 t}{\sin^2 t}\right)$$

$$= 3\cos t\left(\frac{1-\sin^2 t}{\sin^2 t}\right) = 3\cos t\left(\frac{1}{\sin^2 t} - 1\right) = 3\cos t\left(\csc^2 t - 1\right)$$

b. $\tan x - \cot x = \underbrace{\sec}_{\frac{1}{\cos x}} x(2\sin x - \csc x)$

$$\frac{\sin x}{\cos x} - \frac{\cos x}{\sin x} = \frac{1}{\cos x}\left(\sin x - \frac{\cos^2 x}{\sin x}\right) = \frac{1}{\cos x}\left[\sin x - \frac{(1-\sin^2 x)}{\sin x}\right]$$

$$= \frac{1}{\cos x}\left[\sin x - \frac{1}{\sin x} + \frac{\sin^2 x}{\sin x}\right] = \sec(x)\left[2\sin x - \csc x\right]$$

c. $\dfrac{\sec\theta\cdot\sin\theta}{\tan\theta + \cot\theta} = \sin^2\theta$

$$\frac{\frac{1}{\cos\theta}\cdot\sin\theta}{\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}} = \frac{\frac{\sin\theta}{\cos\theta}}{\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}} = \frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\sin\theta\cos\theta}} = \frac{\sin\theta}{\cancel{\cos\theta}}\cdot\sin\theta\,\cancel{\cos\theta}$$

$$= \sin^2\theta$$

2. Find all solutions for:

(a) $\sec(3x) - \cos(3x) = 0$

let $u = 3x$ first Solve $\sec(u) - \cos(u) = 0$

$\frac{1}{\cos u} - \cos u = 0 \rightarrow \frac{1-\cos^2 u}{\cos u} = 0 \Rightarrow 1 - \cos^2 u = 0 \Rightarrow \cos^2 u = 1$

$\cos u = \pm 1$

denom. Note: $\cos u \neq 0$

$\Rightarrow u \neq \frac{\pi}{2} + 2k\pi, \frac{3\pi}{2} + 2k\pi \xrightarrow{u=3x} x \neq \frac{\pi}{6} + \frac{2k\pi}{3}$ & $\pi + \frac{2k\pi}{3}$

general solutions: $\cos u = 1$ or $\cos u = -1$ (k any integer)

$u = 0 + 2k\pi$ $\quad$ $u = \pi + 2k\pi$

$3x = 0 + 2k\pi$ $\quad$ $3x = \pi + 2k\pi$

$x = \frac{2k\pi}{3}$ & $\frac{\pi}{3} + 2k\pi$

(b) $2\sin^2 x = 7\cos x + 5$

$2(1 - \cos^2 x) = 7\cos x + 5 \Rightarrow 2 - 2\cos^2 x = 7\cos x + 5 \Rightarrow 2\cos^2 x + 7\cos x + 3 = 0$

let $u = \cos x$ $\quad 2u^2 + 7u + 3 = 0$

$(2u + 1)(u + 3) = 0$

$u = -\frac{1}{2}$ $\quad$ $u = -3$

$\cos x = -\frac{1}{2}$ $\quad$ $\cos x = -3$ No Solutions.

$x = \frac{2\pi}{3} + 2k\pi, \quad \frac{4\pi}{3} + 2k\pi$

(c) Find all solutions for $4\sin x\cos x = \sqrt{3}$ on $[0, 2\pi)$

$2\sin x\cos x = \frac{\sqrt{3}}{2}$ $\quad \sin(2x) = \frac{\sqrt{3}}{2} \Rightarrow 2x = \frac{\pi}{3} + 2k\pi, \frac{2\pi}{3} + 2k\pi$

General Solutions $\rightarrow x = \frac{\pi}{6} + k\pi, \frac{\pi}{3} + k\pi$

on $[0, 2\pi)$: $\quad 2\pi = \frac{12\pi}{6}$ or $\frac{6\pi}{3}$

let $k=0$: $\frac{\pi}{6}, \frac{\pi}{3}$,

$k=1$: $\frac{7\pi}{6}, \frac{4\pi}{3}$

$K=2$ $\quad \cancel{\frac{13\pi}{6}}, \cancel{\frac{7\pi}{3}}$

(d) Find all solutions for $\dfrac{\cos(2x)}{\cos^2 x} = 1$ on $[0, 2\pi)$

$$\frac{\cos^2 x - \sin^2 x}{\cos^2 x} = 1 \Rightarrow 1 - \frac{\sin^2 x}{\cos^2 x} = 1$$

$$\rightarrow \frac{\sin^2 x}{\cos^2 x} = 0 \Rightarrow \sin^2 x = 0 \Rightarrow \sin x = 0$$

$$x = 0 + 2k\pi, \ \pi + 2k\pi$$

$\cos x \neq 0$

$x \neq \frac{\pi}{2}, \frac{3\pi}{2}$

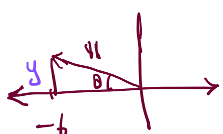
$$x = 2k\pi, \ (2k+1)\pi$$

or $\boxed{x = k\pi}$

on $[0, 2\pi)$: 0 & π

3. Find the exact value for $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$, if $\cos(\theta) = -\dfrac{6}{11}$ and θ is in QII.

$\frac{\text{adj.}}{\text{hyp.}}$



$y^2 + 36 = (11)^2$

$y^2 = 121 - 36$

$y^2 = 85$

$y = \sqrt{85}$

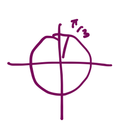
$$\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(\frac{\sqrt{85}}{11}\right)\left(\frac{-6}{11}\right) = \frac{-12\sqrt{85}}{121}$$

$$\cos(2\theta) = 2\cos^2\theta - 1 = 2\left(\frac{-6}{11}\right)^2 - 1 = 2\left(\frac{36}{121}\right) - 1 = \frac{72}{121} - 1 = \frac{72-121}{121} = \frac{-49}{121}$$

$$\tan(2\theta) = \frac{\sin(2\theta)}{\cos(2\theta)} = \frac{-12\sqrt{85}}{-49} = \frac{12\sqrt{85}}{49}$$

4. Find the exact value of $\sin\left(\dfrac{5\pi}{12}\right)$ if $\dfrac{5\pi}{12} = \dfrac{5\pi}{3} - \dfrac{5\pi}{4}$.

$$\sin\left(\frac{5\pi}{3} - \frac{5\pi}{4}\right) = \sin\left(\frac{5\pi}{3}\right)\cos\left(\frac{5\pi}{4}\right) - \cos\left(\frac{5\pi}{3}\right)\sin\left(\frac{5\pi}{4}\right)$$

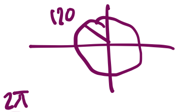


$$= \left(-\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(-\frac{\sqrt{2}}{2}\right)$$

$$= +\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{2}+\sqrt{6}}{4}$$

5. Find the exact value of $\cos(115°)\cos(5°) - \sin(115°)\sin(5°)$.

$\hookrightarrow \cos(115° + 5°) = \cos(120°) = -\frac{1}{2}$



$120° \cdot \frac{\pi}{180°} = \frac{2\pi}{3}$

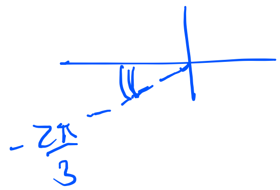
6. Find the exact value of $\dfrac{\tan(\pi/15) + \tan(4\pi/15)}{1 - \tan(\pi/15)\tan(4\pi/15)}$.

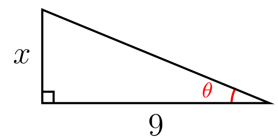
$\Rightarrow \tan\left(\frac{\pi}{15} + \frac{4\pi}{15}\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$

7. Rewrite $\sin(x)\cos(3x) + \sin(3x)\cos(x)$ as a single expression.

$\sin(x + 3x) = \sin(4x)$

8. Use the reference angle to find the value for $\tan\left(-\frac{2\pi}{3}\right) = \tan\left(\pi - \frac{2\pi}{3}\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$





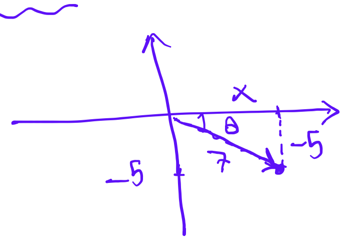
9. Use an inverse trig function to write θ as a function of x.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{\text{opp.}}{\text{hyp.}}}{\frac{\text{adj}}{\text{hyp.}}} = \frac{\text{opp.}}{\text{adj.}} = \frac{x}{9}$$

$$\theta = \arctan\left(\frac{x}{9}\right)$$

10. Given $\csc(\theta) = -\frac{7}{5}$ and $\tan(\theta) < 0$, find the value of $\sec(\theta)$.

$$\frac{\text{hyp.}}{\text{opp.}} = \frac{1}{\sin\theta} = -\frac{7}{5}$$



$$x^2 + 25 = 49$$
$$x = \sqrt{24}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{\text{hyp.}}{\text{adj}} = \frac{7}{\sqrt{24}}$$

11. Emmy chooses a horse that is 10 feet from the center of a merry-go-round. The merry-go-round makes $\frac{9}{2}$ rotations per minute. Determine ~~Jack~~ emmy's angular velocity in radians per second. How far has Emmy travelled in 5 minutes?

1 revolution (or rotation) $\Longleftrightarrow$ 2π radians

angular velocity per second $\omega = \dfrac{\theta}{t} = \dfrac{\frac{9}{2} \text{ rotation/min} \times 2\pi \text{ radians/rotation}}{60 \text{ seconds/min}}$

$= \dfrac{9\pi \text{ radians/min}}{60 \text{ sec/min}} = \dfrac{3\pi}{20}$ radians/second

Distance traveled $=$ radius $\times$ angle (rad)

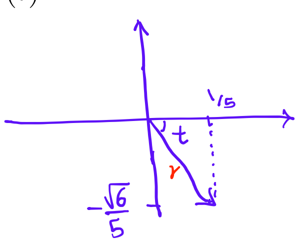
in 5 minutes: $5 \times \frac{9}{2} = \frac{45}{2}$ rotations

angle in radians $\theta = \frac{45}{2} \times 2\pi = 45\pi$ radians

Emmy has traveled?

Distance $= 10 \times 45\pi = 450\pi$ feet in 5 minutes

12. Given t corresponds to the point $\left(\underbrace{\frac{1}{5}}_{x}, \underbrace{-\frac{\sqrt{6}}{5}}_{y}\right)$ ~~on the unit~~ on a circle, find the value of $\sin(t)$, $\sec(t)$, and $\tan(t)$.



$\sin t = \dfrac{\text{opp.}}{\text{hyp}} = \dfrac{-\frac{\sqrt{6}}{5}}{\frac{\sqrt{7}}{5}} = -\sqrt{\frac{6}{7}}$ (or $\sin t = \frac{y}{r} = -\sqrt{\frac{6}{7}}$)

$\cos t = \dfrac{\text{adj.}}{\text{hyp.}} = \dfrac{1/5}{\frac{\sqrt{7}}{5}} = \dfrac{1}{\sqrt{7}}$ (or $\cos t = \frac{x}{r} = \frac{1}{\sqrt{7}}$)

hyp. of triangle (or radius of the circle) $r^2 = \left(\frac{1}{5}\right)^2 + \left(-\frac{\sqrt{6}}{5}\right)^2 = \frac{1}{25} + \frac{6}{25} = \frac{7}{25}$

$r = +\sqrt{7/25} = \dfrac{\sqrt{7}}{5}$

$\sec t = \dfrac{1}{\cos t} = \dfrac{1}{\frac{1}{\sqrt{7}}} = \sqrt{7}$

13. Given $y = \underbrace{\frac{3}{2}}_{A}\sin(5x - \pi) - 2$, state the amplitude, period and phase shift of the graph. Sketch the graph.

start $5x - \pi = 0 \longrightarrow x = \frac{\pi}{5}$ phase shift right

end $5x - \pi = 2\pi \longrightarrow 5x = 3\pi \longrightarrow x = \frac{3\pi}{5}$

one cycle: $\left[\frac{\pi}{5}, \frac{3\pi}{5}\right]$

Period: $\frac{3\pi}{5} - \frac{\pi}{5} = \boxed{\frac{2\pi}{5}}$

Vertical shift 2 down.

$\frac{3}{2} - 2 = -\frac{1}{2}$

$-\frac{3}{2} - 2 = -\frac{7}{2}$

key points

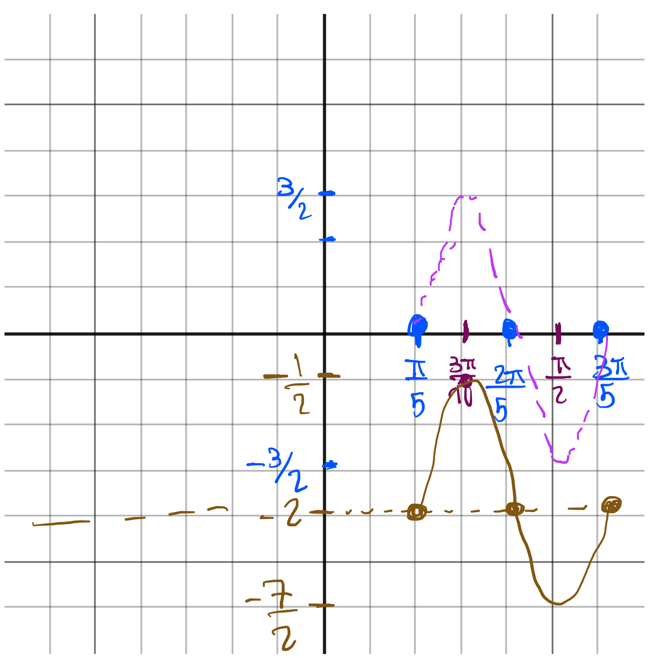
$\theta = \frac{\pi}{5}$ $y = 0$

$\theta = \frac{3\pi}{10}$ $y = 1$

$\theta = \frac{2\pi}{5}$ $y = 0$

$\theta = \frac{5\pi}{10} = \frac{\pi}{2}$ $y = -1$

$\theta = \frac{3\pi}{5}$ $y = 0$



14. State the domain and range of $y = \tan(x)$ $= \dfrac{\sin x}{\cos x}$ $\cos x \neq 0$ $x \neq \frac{\pi}{2} + 2k\pi$, $\frac{3\pi}{2} + 2k\pi$

domain: $\{x \mid x \neq \frac{\pi}{2} + 2n\pi,\ x \neq \frac{3\pi}{2} + 2n\pi\}$

Range $(-\infty, \infty)$

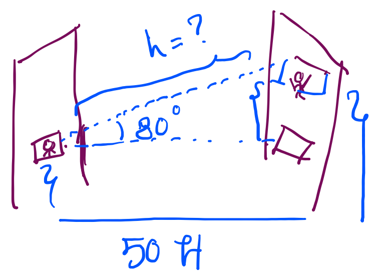
15. State the domain and range of $f(x) = \arcsin(x)$, $g(x) = \arccos(x)$, and $h(x) = \arctan(x)$.

Domain $[-1, 1]$ Domain $[-1, 1]$ Domain $(-\infty, \infty)$

Range $[-\frac{\pi}{2}, \frac{\pi}{2}]$ Range: $[0, \pi]$ Range $[-\frac{\pi}{2}, \frac{\pi}{2}]$

16. From his hotel room window on the sixth floor, Saleh notices some window washers high above him on the hotel across the street. Curious as to their height above the ground, he quickly estimates the buildings are 50 ft apart and the angle of elevation to the workers is 80°. Leave all answers in exact form.

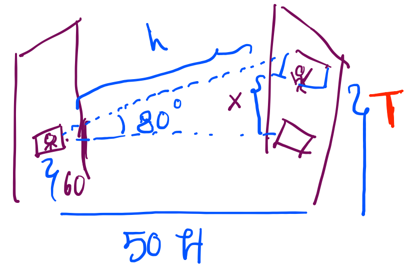
a) How far apart are Saleh and the window washers?



$$\frac{\text{adj.}}{\text{hyp.}} = \frac{50}{h} = \cos(80^\circ)$$

$$h = \frac{50}{\cos(80)}$$

b) If Saleh's hotel floor is 60ft above ground, how far are the window washers from the ground?



$$\sin(80^\circ) = \frac{\text{opp.}}{\text{hyp.}} = \frac{x}{\frac{50}{\cos(80)}} = \frac{x\cos(80)}{50}$$

$$x = 50\,\frac{\sin(80)}{\cos(80)} = 50\tan(80)$$

$$T = x + 60 = 50\tan(80) + 60$$

17. Simplify each composition, if possible.

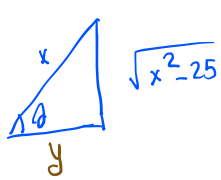
$\tan\left[\arctan(-\sqrt{3})\right] = \tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$

$\arcsin\left[\sin\left(\frac{5\pi}{6}\right)\right] = \arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

$\arcsin\left[\cos(0)\right] = \arcsin(1) = \frac{\pi}{2}$

$\tan\left[\underbrace{\arcsin\left(\frac{\sqrt{x^2-25}}{x}\right)}_{\theta}\right] = \tan(\theta) = \frac{\sin\theta}{\cos\theta} = \frac{\frac{\sqrt{x^2-25}}{x}}{\frac{5}{x}} = \frac{\sqrt{x^2-25}}{5}$

$\arcsin\left(\frac{\sqrt{x^2-25}}{x}\right) = \theta \iff \sin\theta = \frac{\sqrt{x^2-25}}{x} = \frac{\text{opp.}}{\text{hyp.}}$



$\cos\theta = \frac{\text{adj.}}{\text{hyp.}} = \frac{5}{x}$

$y^2 + \left(\sqrt{x^2-25}\right)^2 = x^2$

$y^2 + x^2 - 25 = x^2$

$y^2 = 25 \qquad y = +5$

based on range of arcsin( )

adjacent would always be a + number
(corresponds to x)

18. Evaluate the following

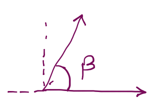
(a) convert 53° to radians.

$$53^\circ = 53(1^\circ) = 53\left(\frac{\pi}{180}\text{rad}\right) = \frac{53\pi}{180}\ \text{rad.}$$

(b) convert $\dfrac{4\pi}{15}$Rad to degrees.

$$\frac{4\pi}{15}\text{rad} = \frac{4\pi}{15}(1\text{rad}) = \frac{4\pi}{15}\left(\frac{180^\circ}{\pi}\right) = (4\times 12)^\circ = 48^\circ$$

(c) Supplementary angle for 53°.



Supplementary angle for α : (means an angle that when added to α, results in 180°) $\Rightarrow$ $180^\circ - 53^\circ = 127^\circ$

(d) Complementary angle for 53°

Complementary angle for β: (an angle that added to β, results in 90°) $\Rightarrow$ $90^\circ - 53^\circ = 37^\circ$

19. solve the following system of equations.

(a) $\begin{cases} (x+4)^2 + y^2 = 4 \\ y - \sqrt{x} = 0 \end{cases}$ $\longrightarrow$ $y = \sqrt{x}$

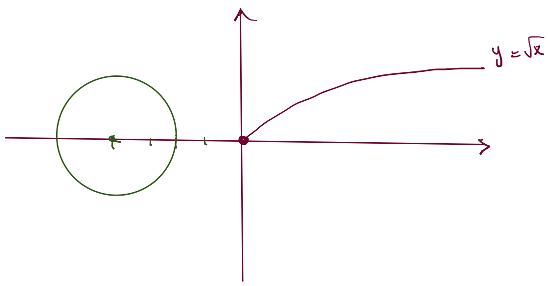
$(x+4)^2 + (\sqrt{x})^2 = 4 \longrightarrow x^2 + 8x + 16 + x = 4$

$x^2 + 9x + 12 = 0$

$x = \frac{-9 - \sqrt{33}}{2} < 0$

$x = \frac{-9 + \sqrt{33}}{2} < 0$

} extraneous since from $\sqrt{x}$ $x \geq 0$



$\Rightarrow$ No solutions!

(b) $\begin{cases} x^2 + y^2 = 25 \\ xy = 12 \end{cases}$ $\rightarrow y = \frac{12}{x}$ $\longrightarrow x^2 + \frac{144}{x^2} = 25$

$x^4 + 144 = 25x^2$

$(x^2)^2 - 25x^2 + 144 = 0$

$u^2 - 25u + 144 = 0$

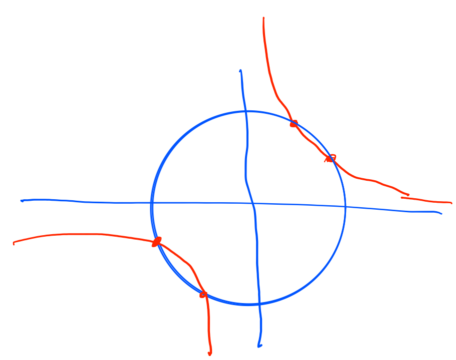
$(u-16)(u-9) = 0 \rightarrow u = 16 \rightarrow x^2 = 16 \rightarrow x = \pm 4$

$u = 9 \rightarrow x^2 = 9 \rightarrow x = \pm 3$

$x = 4 \rightarrow y = 3$
$x = -4 \rightarrow y = -3$
$x = 3 \rightarrow y = 4$
$x = -3 \rightarrow y = -4$

Solutions:

$(4,3), (-4,-3), (3,4), (-3,-4)$



20) Graph $y = 1 + \tan(3x + \frac{\pi}{2})$

extra

Asymptotes : $\cos(3x + \frac{\pi}{2}) = 0$ when $3x + \frac{\pi}{2} = \frac{\pi}{2}$ & $3x + \frac{\pi}{2} = \frac{3\pi}{2}$

$3x = \frac{\pi}{2} - \frac{\pi}{2} \longrightarrow x = 0$

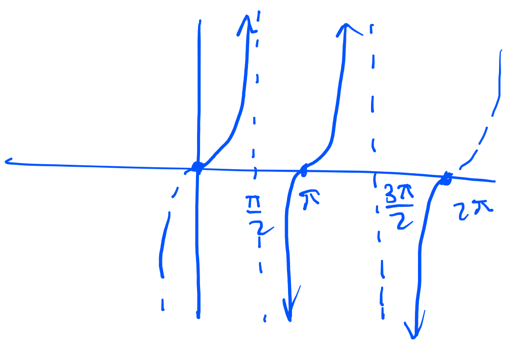
$3x = \frac{3\pi}{2} - \frac{\pi}{2}$ & $x = \frac{\pi}{3}$

starts: $3x + \frac{\pi}{2} = 0$ $x = -\frac{\pi}{6}$ $[-\frac{\pi}{6}, \frac{\pi}{2}]$

Ends: $3x + \frac{\pi}{2} = 2\pi \rightarrow 3x = \frac{3\pi}{2}$ $x = \frac{\pi}{2}$

Period: $\dfrac{\frac{\pi}{2} - (-\frac{\pi}{6})}{2} = \dfrac{\frac{4\pi}{6}}{2} = \frac{\pi}{3}$ (or $\frac{\pi}{B} = \frac{\pi}{3}$)

recall: $\tan x$



$-1 = \tan(3x + \frac{\pi}{2})$

points

x	$y = 1 + \tan(3x + \frac{\pi}{2})$
$-\frac{\pi}{6}$	$y = 1 + \tan(0) = 1$
$\frac{\pi}{6}$	$1 + \tan(\pi) = 1$
$\frac{\pi}{2}$	$1 + \tan(2\pi) = 1$
$\frac{\pi}{12}$	0

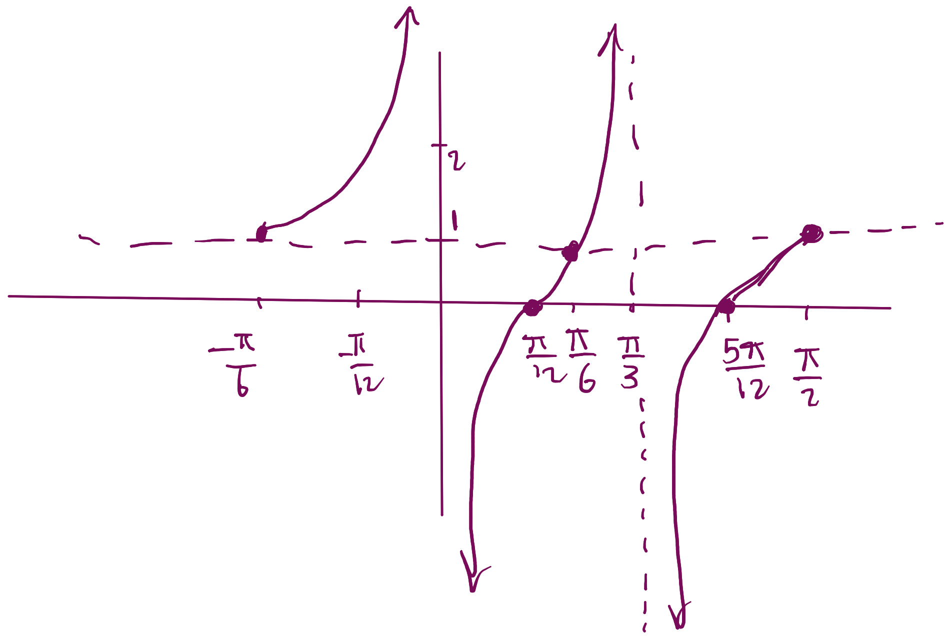
$(\frac{5\pi}{12}, 0)$

$3x + \frac{2\pi}{4} = \frac{3\pi}{4}$

$3x = \frac{\pi}{4}$

$x = \frac{\pi}{12}$

$\frac{7\pi}{4}$



2
1
$-\frac{\pi}{6}$
$-\frac{\pi}{12}$
$\frac{\pi}{12}$
$\frac{\pi}{6}$
$\frac{\pi}{3}$
$\frac{5\pi}{12}$
$\frac{\pi}{2}$