Math 308: Week-in-Review 4

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1. Solve the initial value problems

(a)

$$y'' - 7y' + 12y = 0, \quad y(0) = 3, \ \ y'(0) = -2.$$

* characteristic polynomial : $\lambda^2 - 7\lambda + 12 = 0 \Rightarrow (\lambda-3)(\lambda-4) = 0$

$y(t) = c_1 e^{3t} + c_2 e^{4t}$ (general solution) $\quad \lambda = 3, 4$ two real roots

* Find coefficients c_1 & c_2 using initial conditions

$y(0) = c_1 + c_2 = 3, \quad y'(t) = 3c_1 e^{3t} + 4c_2 e^{4t} \Rightarrow y'(0) = 3c_1 + 4c_2 = -2$

$\Downarrow$

$c_2 = 3 - c_1, \quad 3c_1 + 4(3-c_1) = 3c_1 - 4c_1 + 12 = -2 \Rightarrow c_1 = 14$

$c_2 = 3 - 14 = -11$

$$y(t) = 14e^{3t} - 11e^{4t}$$

(b)

$$y'' + 4y' + 4y = 0, \quad y(0) = 1, \ \ y'(0) = 3.$$

* characteristic polynomial : $\lambda^2 + 4\lambda + 4 = 0 \Rightarrow (\lambda+2)^2 = 0 \Rightarrow \lambda = -2$

$y(t) = c_1 e^{-2t} + c_2 t e^{-2t}$ (general solution) $\quad$ * repeated root *

* Find coefficients of c_1, c_2 using initial conditions

$y(0) = c_1 = 1, \quad y'(t) = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t}$

$y'(0) = -2c_1 + c_2 = 3 \Rightarrow c_2 = 3 + 2c_1 = 3 + 2 = 5$

$$y(t) = e^{-2t} + 5te^{-2t}$$

(c)

$$y'' + 4y' + 20y = 0, \quad y(0) = 3, \quad y'(0) = -1.$$

complex roots

* characteristic polynomial : $\lambda^2 + 4\lambda + 20 = 0 \Rightarrow$

$$\boxed{\begin{aligned}\lambda &= \frac{-4 \pm \sqrt{4^2-(4)(20)}}{2} \\ &= \frac{-4 \pm \sqrt{-64}}{2} \\ &= \frac{-4 \pm 8i}{2} = -2 \pm 4i\end{aligned}}$$

$$y(t) = c_1 e^{-2t}\cos(4t) + c_2 e^{-2t}\sin(4t)$$

* find coefficients c_1, c_2 using initial conds *

$$y(0) = c_1 = 3$$

$$y'(t) = -2c_1 e^{-2t}\cos(4t) - 4c_1 e^{-2t}\sin(4t) - 2c_2 e^{-2t}\sin(4t) + 4c_2 e^{-2t}\cos(4t)$$

$$y'(0) = -2c_1 + 4c_2 = -6 + 4c_2 = -1 \Rightarrow 4c_2 = 5 \Rightarrow c_2 = 5/4$$

$$y(t) = 3e^{-2t}\cos(4t) + \frac{5}{4}e^{-2t}\sin(4t)$$

↙ Inverse problems

2. Find the initial value problems (equations and initial conditions) that have the solutions

(a)

$$y(t) = 2e^{4t} + e^{-6t}$$

* two real roots *
$\lambda_1 = 4$, $\lambda_2 = -6$

* characteristic polynomial : $(\lambda-4)(\lambda+6) = \underbrace{\lambda^2 - 4\lambda + 6\lambda - 24}_{\lambda^2 + 2\lambda - 24} = 0$

* differential equation : $y'' + 2y' - 24y = 0$

* initial conditions : $y(0) = 2 + 1 = 3$,
$y'(t) = 8e^{4t} - 6e^{-6t}$, $y'(0) = 8 - 6 = 2$

$y(t) = 2e^{4t} + e^{-6t}$ is the solution of $y'' + 2y' - 24y = 0$, $y(0) = 3$, $y'(0) = 2$

(b)

$$y(t) = e^{-3t} + 2te^{-3t}$$

* repeated root * $\lambda = -3$

* characteristic polynomial: $(\lambda+3)^2 = \lambda^2+6\lambda+9=0$

* differential equation: $y''+6y'+9y=0$

* initial conditions: $y(0) = 1+0 = 1$

$y'(t) = -3e^{-3t} + 2e^{-3t} - 6te^{-3t}$

$y'(0) = -3+2 = -1$

$y(t) = e^{-3t} + 2te^{-3t}$ is the solution of $y''+6y'+9y=0$, $y(0)=1$, $y'(0)=-1$

(c)

$$y(t) = 2e^{-t/2}\cos(3t) + e^{-t/2}\sin(3t)$$

* complex roots *

$\lambda = -\frac{1}{2} \pm 3i$

$= \frac{-1\pm 6i}{2}$

* characteristic polynomial: $\lambda^2+a\lambda+b=0$

$\lambda = \frac{-1\pm 6i}{2} \Rightarrow a=1$

$\sqrt{1^2-4b} = 6i \Leftrightarrow \sqrt{4b-1} = 6$

$4b-1 = 36 \Rightarrow 4b = 37 \Rightarrow b = \frac{37}{4}$

$\lambda^2+\lambda+\frac{37}{4} = 0$

$\Updownarrow$

$4\lambda^2+4\lambda+37=0$

* differential equation: $4y''+4y'+37=0$

* initial conditions: $y(0)=2$, $y'(t) = -e^{-t/2}\cos(3t) - 6e^{-t/2}\sin(3t) - \frac{1}{2}e^{-t/2}\sin(3t) + 3e^{-t/2}\cos(3t)$

$y'(0) = -1+3 = 2$

$y(t) = 2e^{-t/2}\cos(3t) + e^{-t/2}\sin(3t)$ is the solution of $4y''+4y'+37=0$, $y(0)=2$, $y'(0)=2$

3. Verify that $y_1 = \cos(\ln x^2)$ and $y_2 = \sin(\ln x^2)$ are solutions of the differential equation

$$x^2y'' + xy' + 4y = 0. \quad , \quad x > 0.$$

Do they constitute a fundamental set?

$y_1 = \cos(\ln x^2)$, $y_1' = -\sin(\ln x^2)\cdot\frac{1}{x^2}\cdot 2x = -\frac{2\sin(\ln x^2)}{x}$ * by the chain rule

$$y_1'' = \frac{x\cdot\left[-2\cos(\ln x^2)\cdot\frac{1}{x^2}\cdot 2x\right] - \left[-2\sin(\ln x^2)\right]}{x^2}$$

$$= -\frac{4\cos(\ln x^2) + 2\sin(\ln x^2)}{x^2}$$

$$x^2y_1'' + xy_1' + 4y_1 = \left[-4\cancel{\cos(\ln x^2)} + 2\cancel{\sin(\ln x^2)}\right] + \left[-2\cancel{\sin(\ln x^2)}\right] + 4\cancel{\cos(\ln x^2)}$$

$= 0$ ✓ * y_1 is a solution

$y_2 = \sin(\ln x^2)$, $y_2' = \cos(\ln x^2)\cdot\frac{1}{x^2}\cdot 2x = \frac{2\cos(\ln x^2)}{x}$

$$y_2'' = \frac{x\cdot\left[-2\sin(\ln x^2)\cdot\frac{1}{x^2}\cdot 2x\right] - 2\cos(\ln x^2)}{x^2}$$

$$= -\frac{4\sin(\ln x^2) - 2\cos(\ln x^2)}{x^2}$$

$$x^2y_2'' + xy_2' + 4y_2 = \left[-4\cancel{\sin(\ln x^2)} - 2\cancel{\cos(\ln x^2)}\right] + 2\cancel{\cos(\ln x^2)} + 4\cancel{\sin(\ln x^2)}$$

$= 0$ ✓ * y_2 is a solution

* check for linear independence (y_1, y_2 form a fundamental pair?)

$$W(t) = y_1y_2' - y_1'y_2 = \cos(\ln x^2)\left[\frac{2\cos(\ln x^2)}{x}\right] - \left[-\frac{2\sin(\ln x^2)}{x}\right](\sin(\ln x^2))$$

$$= \frac{2}{x}\left[\cos^2(\ln x^2) + \sin^2(\ln x^2)\right] = 2/x \neq 0, \quad x > 0$$

✓

Therefore $\{\cos(\ln x^2), \sin(\ln x^2)\}$ is a fundamental pair.

4. Suppose $y_1 = t^{-1/2}$ is a solution of the differential equation

$$4t^2y'' + 8ty' + y = 0,\ t > 0.$$

Determine a second linearly independent solution y_2.

* Method of reduction of order * Set $y_2 = u(t)\,t^{-1/2}$

Then, $y_2' = u't^{-1/2} - \frac{1}{2}ut^{-3/2}$ (product rule)

$$y_2'' = u''t^{-1/2} - \frac{1}{2}u't^{-3/2} - \frac{1}{2}u't^{-3/2} + \frac{3}{4}ut^{-5/2} = u''t^{-1/2} - u't^{-3/2} + \frac{3}{4}ut^{-5/2}$$

$$4t^2y_2'' + 8ty_2' + y_2 = 4t^2\cdot u''\cdot t^{-1/2} - 4t^2\cdot u'\cdot t^{-3/2} + 4t^2\cdot\frac{3}{4}\cdot u\cdot t^{-5/2}$$

plug y_2 into equation

$$+ 8t\cdot u'\cdot t^{-1/2} - 8t\cdot\frac{1}{2}\cdot u\cdot t^{-3/2} + ut^{-1/2}$$

$$= 4t^{3/2}u'' - 4t^{1/2}u' + \cancel{3t^{-1/2}}u + 8t^{1/2}u' - \cancel{4t^{-1/2}}u + \cancel{ut^{-1/2}}$$

$$= 4t^{3/2}u'' + 4t^{1/2}u' = 0 \quad \text{(since } y_2 \text{ is a solution)}$$

$$4t^{3/2}u'' + 4t^{1/2}u' = 0 \Leftrightarrow u'' + \frac{1}{t}u' = 0 \quad \text{(solve for } u\text{)}$$

Set $w = u' \Rightarrow w' = u'' \Rightarrow w' + \frac{1}{t}w = 0$ $p(t) = \frac{1}{t} \Rightarrow \mu(t) = e^{\int \frac{1}{t}dt} = e^{\ln t} = t$

$$\underbrace{tw' + w}_{} = 0$$

$$(tw)' = 0 \Rightarrow tw = c \Rightarrow w = c/t \Rightarrow u' = c/t \Rightarrow u = c\int 1/t\,dt = \underset{\text{arbitrary}}{c}\ln t$$

if $c = 1$, then $u = \ln(t)$. Therefore $y_2 = t^{-1/2}\ln t$

$$y_2 = \frac{\ln(t)}{\sqrt{t}}$$

General solution: $y(t) = \dfrac{c_1}{\sqrt{t}} + \dfrac{c_2\ln(t)}{\sqrt{t}}$

5. If the differential equation

$$3t^2y'' - 2ty' - 5y = 0, \ t > 0$$

has a fundamental set of solutions y_1 and y_2 and $W(y_1, y_2)(1) = 5$, find the value of $W(y_1, y_2)(8)$.

* rewrite in standard form * $\quad y'' - \frac{2}{3t}y' - \frac{5}{3t^2}y = 0$

$p(t) = -\frac{2}{3t}$

* find Wronskian using Abel's formula *

$$W[y_1,y_2](t) \triangleq W(t) = k\,e^{\int -p(t)dt} = k\,e^{\int \frac{2}{3t}dt} = k\,e^{\frac{2}{3}\ln t} = k\,t^{2/3}$$

$$W(1) = k = 5$$

$$W(t) = 5t^{2/3}$$

$$W(8) = 5\left(8^{2/3}\right) = 5\left(\sqrt[3]{8}\right)^2$$

$$= 5\left(2^2\right)$$

$$= 20$$

6. Find a general solution of

$$4t^2y'' + 4ty' + (36t^2 - 1)y = 0$$

given that $y = t^{-1/2}\cos(3t)$ is one solution.

* Set $y_1 = t^{-1/2}\cos(3t)$ and use method of reduction of order to find y_2.

$$y_2 = u(t)\,t^{-1/2}\cos(3t)$$

* We use Sympy to evaluate the condition on u needed to satisfy the differential equation

```
from sympy import *

t = symbols('t')         # declare symbol t
y1 = Function('y1')(t)  # declare function y1
y2 = Function('y2')(t)  # declare function y2
u = Function('u')(t)    # declare function u
y1 = cos(3*t)/sqrt(t)   # define y1
y2 = u*y1               # define y2
Q = 4*t**2*y2.diff(t,2)+4*t*y2.diff(t)+(36*t**2-1)*y2 # plug y2 into equation
simplify(Q)   # simplify expression
```

* Sympy output : $4t^2y_2'' + 4ty_2' + (36t^2-1)y_2 = 4t^{3/2}\left(-6u'\sin(3t) + u''\cos(3t)\right) = 0$

* Solve for u: $u'' - 6\tan(3t)u' = 0$, $w = u'$, $w' = u''$

$w' - 6\tan(3t)w = 0$, $p(t) = -6\tan(3t)$,

$$\boxed{\begin{aligned}\mu(t) &= e^{\int p(t)dt} \\ &= e^{-6\int \tan(3t)dt} \\ &= e^{2\ln\cos(3t)} \\ &= \cos^2(3t)\end{aligned}}$$

$\cos^2(3t)w' - 6\tan(3t)\cos^2(3t)w = 0$

$\cos^2(3t)w' - 6\sin(3t)\cos(3t)w = 0$

$(\cos^2(3t)w)' = 0 \Rightarrow \cos^2(3t)w = c$

$w = \dfrac{c}{\cos^2(3t)} = c\sec^2(3t)$ $c = 1$

$u' = \sec^2(3t) \Rightarrow u = \int \sec^2(3t)dt = \frac{1}{3}\tan(3t)$

$y_2 = \frac{1}{3}\tan(3t)\cdot t^{-1/2}\cdot\cos(3t)$

$= \frac{1}{3}t^{-1/2}\sin(3t)$

$$\Rightarrow \boxed{\begin{aligned}y(t) &= c_1t^{-1/2}\cos(3t) + \left(c_2 + \tfrac{1}{3}\right)t^{-1/2}\sin(3t) \\ y(t) &= C_1t^{-1/2}\cos(3t) + C_2t^{-1/2}\sin(3t)\end{aligned}}$$

General Solution