Math 308: Week-in-Review 2

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1. Find the general solution of the given differential equations

   (a) $y' + 2ty = 2te^{-t^2}$

first order linear : $y' + p(t)y = g(t)$

$p(t) = 2t$, $\mu(t) = e^{\int p(t)dt} = e^{\int 2t dt} = e^{t^2}$

$\mu(t)$ integrating factor

$\underbrace{e^{t^2} y' + 2te^{t^2} y}_{} = 2t \cdot \cancel{e^{-t^2}} \cdot \cancel{e^{t^2}} = 2t$

$(e^{t^2} y)' = 2t$

$e^{t^2} y = \int 2t\, dt = t^2 + C$

$$\boxed{y = t^2 e^{-t^2} + Ce^{-t^2}}$$

nonlinear, separable

   (b) $2\sqrt{x}y' = \sqrt{1-y^2}$

Equilibrium solutions : $\sqrt{1-y^2} = 0 \Rightarrow 1 - y^2 = 0 \Rightarrow y = \pm 1$

$\frac{1}{\sqrt{1-y^2}} dy = \frac{1}{2\sqrt{x}} dx$

$\int \frac{1}{\sqrt{1-y^2}} dy = \frac{1}{2} \int \frac{1}{\sqrt{x}} dx$

$\arcsin(y) = \frac{1}{\cancel{2}} \cdot \cancel{2}\sqrt{x} + C = \sqrt{x} + C$, or $\sin^{-1}(y) = \sqrt{x} + C$

$$\boxed{y = \sin(\sqrt{x} + C)}$$

(c) $ty' + y = 3t\cos t, \qquad t > 0$ first order linear :

$(ty)' = 3t\cos t$

$ty = \underbrace{\int 3t\cos t\,dt}_{\text{integrate by parts}} = 3t\sin t + 3\cos t + C$

$\int \underbrace{3t}_{u}\,\underbrace{\cos t\,dt}_{dv} = 3t\sin t - \int \sin t \cdot 3\,dt$
$= 3t\sin t + 3\cos t + C$

$\int u\,dv = uv - \int v\,du$

$\begin{cases} u = 3t, & du = 3\,dt \\ dv = \cos t\,dt, & v = \sin t \end{cases}$

$y = 3\sin t + 3\dfrac{\cos t}{t} + \dfrac{C}{t}$

2. Find the solution to the initial value problem and the interval of validity in each case

(a) $2\sqrt{x}\dfrac{dy}{dx} = \cos^2(y), \qquad y(4) = \dfrac{\pi}{4}$ nonlinear, separable

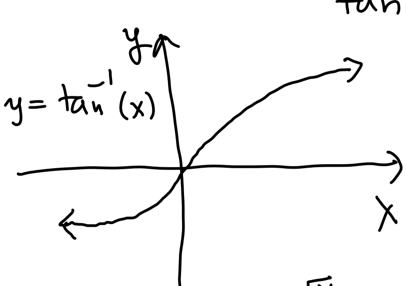
Equilibrium solutions :
$\cos^2(y) = 0 \Rightarrow \cos(y) = 0$
$y = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots$
none of these are $\pi/4$

$\dfrac{1}{\cos^2(y)}\,dy = \dfrac{1}{2\sqrt{x}}\,dx$

$\int \sec^2(y)\,dy = \frac{1}{2}\int \frac{1}{\sqrt{x}}\,dx$

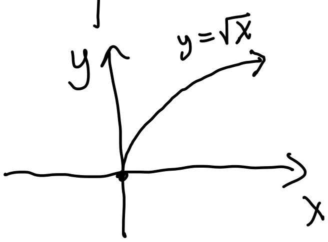
$\tan(y) = \cancel{\tfrac{1}{2}} \cdot \cancel{2} \cdot \sqrt{x} + C$

$\tan(y) = \sqrt{x} + C, \quad \tan\left(\frac{\pi}{4}\right) = \sqrt{4} + C \Rightarrow 1 = 2 + C$
$C = -1$



$y = \underbrace{\arctan}_{\text{defined } (-\infty,\infty)}(\sqrt{x} - 1)$ or $y = \tan^{-1}(\sqrt{x} - 1)$

$\sqrt{x}$ differentiable on $(0, \infty)$



Interval of validity: $(0, \infty)$

(b) $\dfrac{dy}{dt} + \dfrac{2y}{t} = \dfrac{\cos t}{t^2}$, $\quad y(1) = \dfrac{1}{2}$, $\quad t > 0$ first order linear

$p(t) = 2/t \Rightarrow \mu(t) = e^{\int 2/t\, dt} = e^{2\ln t} = e^{\ln t^2} = t^2$ integrating factor

$t^2 y' + \cancel{t^2}\dfrac{2}{\cancel{t}}y = \cancel{t^2}\cdot\dfrac{\cos t}{\cancel{t^2}} \Rightarrow t^2y' + 2ty = \cos t$

$(t^2 y)' = \cos t$

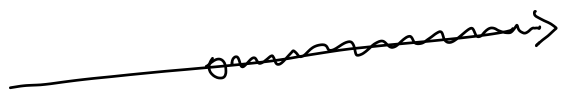
$t^2 y = \int \cos t\, dt = \sin t + C$

initial condition: $t = 1,\ y = \frac{1}{2}$

$1^2 \cdot \frac{1}{2} = \sin(1) + C \Rightarrow C = \frac{1}{2} - \sin(1)$

$$\boxed{y = \frac{\sin t}{t^2} + \frac{\left[\frac{1}{2} - \sin(1)\right]}{t^2}}$$

Interval of validity: $t > 0$



$t = 0$

3. Find the value of y_0 for which the solution of the initial value problem

$$y' - y = 1 + 2\sin(t), \quad y(0) = y_0$$

first order linear

remains finite as $t \to \infty$

$p(t) = -1$, $\mu(t) = e^{\int -1\,dt} = e^{-t}$ integrating factor

$$\underbrace{e^{-t}y' - e^{-t}y}_{} = e^{-t} + 2e^{-t}\sin t$$

$$(e^{-t}y)' = e^{-t} + 2e^{-t}\sin t$$

integrate by parts

$$e^{-t}y = \int e^{-t}\,dt + 2\int e^{-t}\sin t\,dt$$

$$= -e^{-t} + 2I + c, \text{ where } I = \int e^{-t}\sin t\,dt$$

$\int u\,dv = uv - \int v\,du$

$u = \sin t$, $du = \cos t\,dt$

$dv = e^{-t}\,dt$, $v = -e^{-t}$

$$I = \int e^{-t}\sin t\,dt = -e^{-t}\sin t + \int e^{-t}\cos t\,dt$$

integrate by parts

$u = \cos t$, $du = -\sin t\,dt$

$dv = e^{-t}\,dt$, $v = -e^{-t}$

$$= -e^{-t}\sin t - e^{-t}\cos t - \int e^{-t}\sin t\,dt$$

$$I = -e^{-t}\sin t - e^{-t}\cos t - I$$

$$2I = -e^{-t}(\sin t + \cos t)$$

$y(0) = -1 - 1 + C = y_0$

$C = y_0 + 2$

$$e^{-t}y = -e^{-t} - e^{-t}(\sin t + \cos t) + C$$

$$y = -1 - (\sin t + \cos t) + Ce^{t}$$

$$y = \underbrace{-1}_{\text{bounded}} - \underbrace{(\sin t + \cos t)}_{\text{bounded}} + \underbrace{(y_0 + 2)e^{t}}_{\text{unbounded unless } y_0 + 2 = 0}$$

The function remains finite as $t \to \infty$ if $\boxed{y_0 = -2}$

4. A 120 gallon tank initially contains 90 pounds of salt dissolved in 90 gallons of water. Brine containing 2 pounds per gallon of salt flows into the tank at a rate of 4 gallons per minute, and the well-stirred mixture flows out of the tank at a rate of 3 gallons per minute. How much salt does the tank contain when it is full?

$Q(t) \to$ amount of salt at time t in lbs

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out} = 2 * 4 - \frac{Q(t)}{V(t)} * 3$$

$\to$ flow rate out

2: concentration; 4: flow rate in; $V(t)$: volume

$$V(t) = 90 + (4-3)t = 90 + t$$

$$\left\{ \frac{dQ}{dt} = 8 - \frac{3Q}{90+t}, \quad Q(0) = 90 \right\} \Rightarrow Q'(t) + \frac{3}{90+t} Q(t) = 8$$

first order linear

$$p(t) = \frac{3}{90+t} \Rightarrow \mu(t) = e^{3\int \frac{1}{90+t} dt} = e^{3\ln(90+t)} = e^{\ln(90+t)^3}$$

$$\boxed{\mu(t) = (90+t)^3}$$

$$(90+t)^3 Q'(t) + (90+t)^3 \cdot \frac{3}{90+t} Q(t) = 8(90+t)^3$$

$$\left((90+t)^3 Q\right)' = 8(90+t)^3$$

$$(90+t)^3 Q = 8\int (90+t)^3 dt = \frac{8}{4}(90+t)^4 + C = 2(90+t)^4 + C$$

$$Q(t) = 2(90+t) + C(90+t)^{-3}$$

$$Q(0) = 180 + C \cdot 90^{-3} = 90 \to \text{initial condition}$$

$$C = -90^4$$

$$\boxed{Q(t) = 180 + 2t - 90^4(90+t)^{-3}}$$

Tank is full when $V(t) = 120$
i.e. $t = 30$

$$Q(30) = 180 + 2(30) - 90^4(120)^{-3}$$

$$\boxed{Q(30) \simeq 202 \text{ lbs}}$$

5. Suppose you deposit \$6000 in an account that accrues interest continuously. Assuming no deposits and withdrawals, how much will be in the account after six years if the interest rate is a constant at 4.5% for the first three years and 5.4% for the last three years?

$A(t) \to$ amount in account at time t, $A_0 \to$ initial deposit

$$\begin{cases} A'(t) = r\,A(t) \\ A(0) = A_0 \end{cases} \qquad r \Rightarrow \text{annual interest rate}$$

separable

$$\frac{1}{A(t)} A'(t) = r$$

$$\frac{d}{dt} \ln |A(t)| = r$$

$$\ln |A(t)| = rt + c$$

$$|A(t)| = e^{c} \cdot e^{rt}$$

$$\to A(t) = \pm e^{c} \cdot e^{rt} = C e^{rt}, \quad C = \pm e^{c}$$

$$A(0) = C = A_0$$

$$\boxed{A(t) = A_0 e^{rt}}$$

$$A(t) = \begin{cases} 6000\, e^{0.045t}, & \text{if } 0 \leq t < 3 \\ A_1 e^{0.054(t-3)}, & \text{if } 3 \leq t \leq 6 \end{cases}$$

where A_1 is the amount after 3 years, $A_1 = 6000 e^{0.045 * 3}$
$= \$6867.22$

Amount after 6 years, $A(6) = 6867.22\, e^{0.054(6-3)}$

$= 6867.22\, e^{0.054 * 3}$

$$\boxed{A(6) = \$\,8{,}074.89}$$

6. A population of rats at a corn farm increases at a rate proportional to their current population, and in the absence of other factors (no predators for example), the farmer notices that the rat population doubles in one month. There are 300 rats at the farm initially, and the farmer releases weasels that eat 100 rats per month. $y(t) \to$ population of rats at time t

Find k: Population doubles in a month

$y'(t) = k\,y(t)$

$y(t) = y_0 e^{kt}$

$2\cancel{y_0} = \cancel{y_0} e^{k} \Rightarrow k = \ln(2)$

   (a) Determine the population of rats at the farm at any time.

$y'(t) = K\,y(t) - h$

$K \to$ growth constant

$h \to$ rate of predation $= 100$ per month

$y' = \ln(2)y - 100, \quad y(0) = 300$ first order linear

$y' - \ln(2)y = -100, \quad p(t) = -\ln(2) \Rightarrow \mu(t) = e^{-\int \ln(2)dt} = e^{-t\ln(2)} = e^{\ln(2^{-t})} = 2^{-t}$

$2^{-t}y' - 2^{-t}\ln(2)y = -100 \cdot 2^{-t}$

$(2^{-t}y)' = -100 \cdot 2^{-t}$

$2^{-t}y = -100\int 2^{-t}dt = \frac{100}{\ln(2)} \cdot 2^{-t} + C$

$y(0) = 300 = \frac{100}{\ln(2)} + C \Rightarrow C = 300 - \frac{100}{\ln(2)}$

$y(t) = \cancel{2^{t}} \cdot \frac{100}{\ln(2)} \cdot \cancel{2^{-t}} + \left(300 - \frac{100}{\ln(2)}\right) \cdot 2^{t}$

$$\boxed{y(t) = 300 \cdot 2^{t} - \frac{100}{\ln(2)}\left[2^{t} - 1\right]}$$

   (b) Find the time for the population of rats to double to 600 in the presence of weasels.

Find t when $y = 600$

$600 = 300 \cdot 2^{t} - \frac{100}{\ln(2)} \cdot 2^{t} + \frac{100}{\ln(2)} \Rightarrow 2^{t}\left(300 - \frac{100}{\ln(2)}\right) = 600 - \frac{100}{\ln(2)}$

$2^{t} = \frac{\left[600 - \frac{100}{\ln 2}\right]}{\left[300 - \frac{100}{\ln 2}\right]} \Rightarrow t = \log_2\left(\frac{600 - \frac{100}{\ln(2)}}{300 - \frac{100}{\ln 2}}\right)$

$$\boxed{t = 1.549 \text{ months}}$$

up is +ve

$\uparrow \vec{v}$

$\downarrow mg$ (gravity) $\quad \downarrow v/10$ $\leftrightarrow$ air resistance

7. A ball with mass 1 kg is thrown upward with initial velocity 20 m/s from the roof of a building 50 m high. A force due to air resistance of $v/10$ where the velocity is measured in m/s acts on the ball. Find the maximum height above the ground that the ball reaches. v = velocity, m = mass, $g = 9.81\ m/s^2$

$F = ma$

$$-mg - \frac{v}{10} = m\frac{dv}{dt} \Rightarrow \begin{cases} \frac{dv}{dt} + \frac{v}{10} = -g \quad \text{since } m = 1\text{ kg} \\ v(0) = 20\ m/s \end{cases}$$

first order linear

$$p(t) = \frac{1}{10} \Rightarrow \mu(t) = e^{\int \frac{1}{10}dt} = e^{\frac{1}{10}t}$$

$$e^{\frac{1}{10}t}v' + e^{\frac{1}{10}t}v = -ge^{\frac{1}{10}t}$$

$$\left(e^{\frac{1}{10}t}v\right)' = -ge^{\frac{1}{10}t}$$

$$e^{\frac{1}{10}t}v = -g\int e^{\frac{1}{10}t}dt = -10ge^{\frac{1}{10}t} + C$$

$$v(t) = -10g\cancel{e^{\frac{1}{10}t}} \cdot \cancel{e^{-\frac{1}{10}t}} + Ce^{-\frac{1}{10}t}$$

$$v = -10g + Ce^{-\frac{1}{10}t}$$

$$v(0) = 20 = -10g + C \Rightarrow C = 20 + 10g$$

$$\boxed{v(t) = -10g + (10g+20)e^{-\frac{1}{10}t}}$$

At maximum height, $v(t_m) = 0$, $t_m \Rightarrow$ time @ max height

$$0 = -10g + (10g+20)e^{-\frac{1}{10}t_m} \Rightarrow e^{-\frac{1}{10}t_m} = \frac{10g}{10g+20}$$

$$-\frac{t_m}{10} = \ln\left(\frac{10g}{10g+20}\right) \Rightarrow t_m = 10\ln\left(\frac{10g+20}{10g}\right) = 10\ln\left(1+\frac{2}{g}\right) \Rightarrow \boxed{t_m = 10\ln\left(1+\frac{2}{g}\right)}$$

H = max height above ground $= h + 50\,m$

$$h = \int_0^{t_m} v(t)\,dt$$

$$= \int_0^{10\ln(1+2/g)} \left(-10g + (10g+20)e^{-\frac{1}{10}t}\right)dt$$

$$= -10g\left(10\ln\left(1+\frac{2}{g}\right)\right) - 10(10g+20)e^{-\frac{1}{10}t}\Big|_0^{10\ln(1+2/g)}$$

$$= 100(g+2)\left[1 - e^{-\ln(1+2/g)}\right] - 100g\ln\left(1+\frac{2}{g}\right)$$

$$= 100(g+2)\left[1 - \frac{g}{g+2}\right] - 10g\left(10\ln\left(1+\frac{2}{g}\right)\right)$$

$\frac{2}{g+2}$

$$h = 200 - 100g\ln\left(1+\frac{2}{g}\right) \approx 17.98\ m$$

$$H = h + 50 = 67.98\ m$$