



Problem 1. Analyze the function $f(x) = (x^2 - 1)(x^2 - 7)$ → polynomial of order 4

(1) What is the Domain of the function?

$$D: (-\infty, \infty)$$

$$\begin{aligned} f(-2) &= (4-1)(4-7) \\ &= 3(-3) \\ &= -9 \end{aligned}$$

Rules of Domains

- ① No division by zero.
- ② If $\sqrt[n]{x}$, n is even,
- ③ $\log_a(x)$, $x > 0$

(2) What are the x -intercept(s) and what is the y -intercept of the function?

$$\begin{aligned} (x^2 - 1)(x^2 - 7) &= 0 \\ x^2 - 1 &= 0 \quad x^2 - 7 = 0 \\ x = \pm 1 & \quad x = \pm\sqrt{7} \end{aligned}$$

$y = 0$

$x = 0$, $f(0) = (0-1)(0-7) = +7$

$(\pm 1, 0)$

$(\pm\sqrt{7}, 0)$ → $(\pm 2.64, 0)$

$(0, 7)$

$x = \pm 1$ (3) Does the function have any vertical or horizontal asymptotes?

no division by zero

∴ none

as $x \rightarrow \pm\infty$

$x^4 \rightarrow \infty$

∴ none

(4) What are the critical points of the function?

$$\begin{aligned} f'(x) &= 0 \quad \text{or} \quad f'(x) \text{ DNE} \\ f(x) &= (x^2 - 1)(x^2 - 7) \\ f'(x) &= (x^2 - 1)(2x) + (x^2 - 7)(2x) \end{aligned}$$

$f'(x) = 2x(x^2 - 1 + x^2 - 7)$

$\Rightarrow f'(x) = 2x(2x^2 - 8) = 0$

$x = 0$

$2x^2 - 8 = 0$

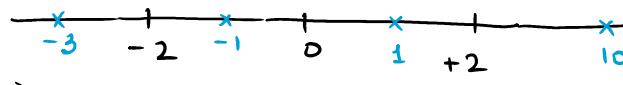
$x^2 = \frac{8}{2} = 4$

$x = \pm 2$

(5) On which interval(s) is the function increasing and decreasing?

$$f'(x) = 2x(2x^2 - 8)$$

$$(-)(+) = (-) \quad (-)(-) = (+) \quad (+)(-) = (-) \quad (+)(+) = (+)$$



$f(x)$:

$f(x)$ is decreasing on $(-\infty, -2) \cup (0, 2)$

$f(x)$ is increasing on $(-2, 0) \cup (2, \infty)$

$$IPs: \left(-\frac{2}{\sqrt{3}}, -\frac{17}{9}\right), \left(\frac{2}{\sqrt{3}}, -\frac{17}{9}\right)$$

2 $f''(x) = 0$ or $f''(x)$ DNE

(6) Find the inflection points of the function.

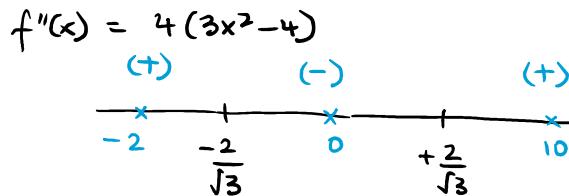
$$f'(x) = 2x(2x^2 - 8) = 4x(x^2 - 4)$$

$$\begin{aligned} f''(x) &= (4x)(2x) + (x^2 - 4)(4) \\ &= 8x^2 + 4x^2 - 16 \end{aligned}$$

$$f''(x) = 12x^2 - 16 = 0$$

$$\begin{aligned} 12x^2 &= 16 \\ x^2 &= \frac{16}{12} = \frac{4}{3} \\ x &= \pm\sqrt{\frac{4}{3}} = \pm\frac{2}{\sqrt{3}} \\ &\approx \pm 1.15 \end{aligned}$$

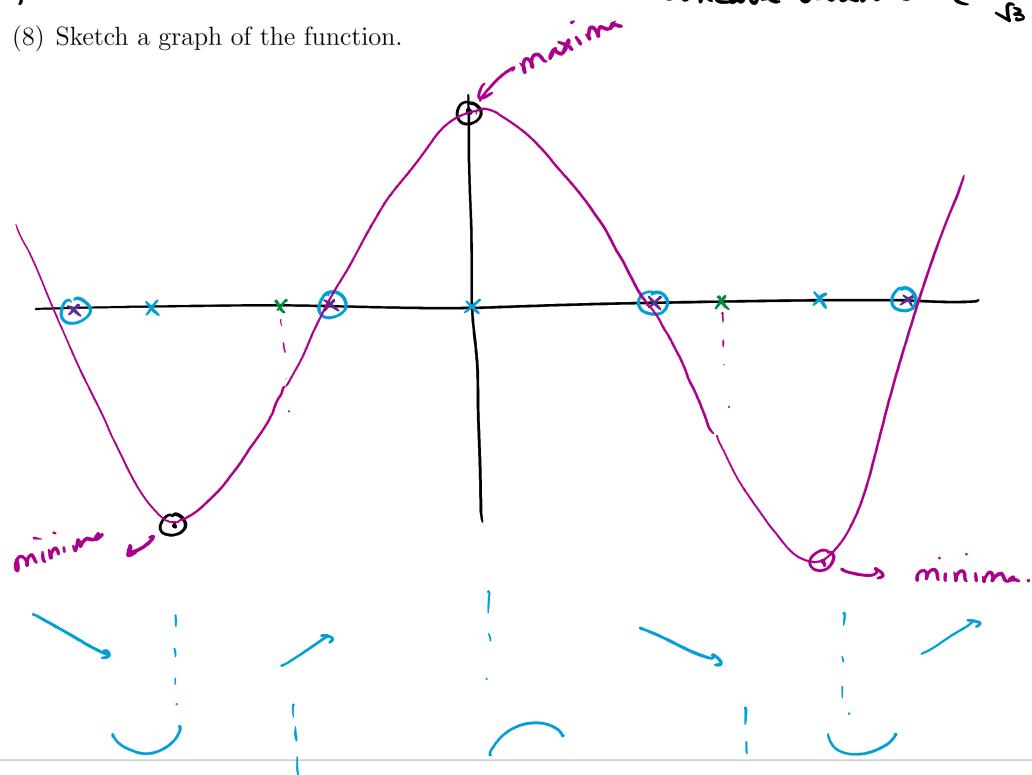
(7) On which interval(s) is the function concave up? Concave down?



$f(x)$ is
concave up on $(-\infty, -\frac{2}{\sqrt{3}}) \cup (\frac{2}{\sqrt{3}}, \infty)$

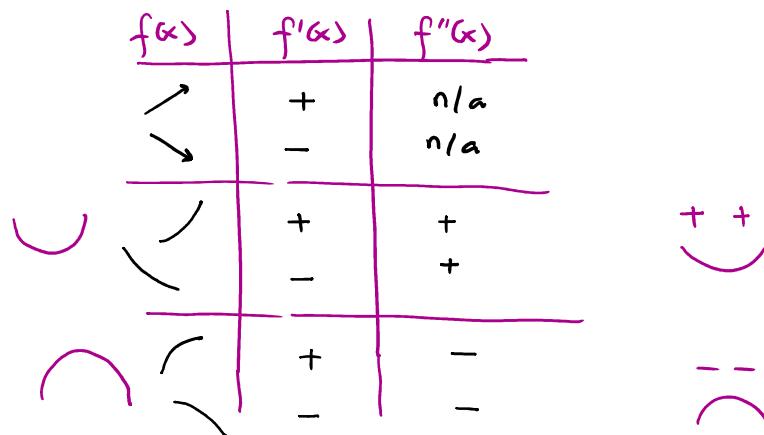
concave down on $(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}})$

(8) Sketch a graph of the function.

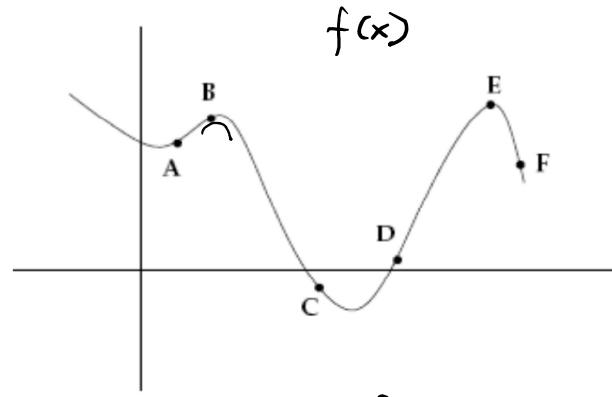


Problem 2. Discuss the relation between f , f' and f'' .

$f(x)$	$f'(x)$	$f''(x)$
↑	+	n/a
↓	-	n/a
↑	+	+
↓	-	+
↑	+	-
↓	-	-



Problem 3. At what points on the graph of $f(x)$ given below, does $f'(x)$ have the same sign as $f''(x)$? (Thanks to J. Kahlig)



	$f'(x)$	$f''(x)$
A	+	+
B	+	-
C	-	+
D	+	+
E	0	-
F	-	-

Problem 4. Find the derivative of $f(x) = \left(\frac{6x}{5^x - 1}\right)^4$

$$f'(x) = 4\left(\frac{6x}{5^x - 1}\right)^3 \cdot \frac{d}{dx}\left(\frac{6x}{5^x - 1}\right)$$

$$= 4\left(\frac{6x}{5^x - 1}\right)^3 \cdot \left[\frac{(5^x - 1)(6) - (6x)(5^x \ln 5)}{(5^x - 1)^2} \right]$$

Problem 5. Find the derivative of $f(x) = (4x - \sqrt{5-x})^{1/2} (\log_2 x)^3$

$$f'(x) = (4x - \sqrt{5-x}) \left[3(\log_2 x)^2 \cdot \underbrace{\frac{d}{dx}(\log_2 x)}_{x \cdot (\ln 2)} \right] + (\log_2 x)^3 \left[4 - \frac{1}{2}(5-x)^{-1/2} \cdot \underbrace{\frac{d}{dx}(5-x)}_{(c-1)} \right]$$

Problem 6. Find the value of a such that the tangent line to the curve $y = \ln x$ at $x = a$ passes through the origin.

(0,0)

$$y = \ln(x)$$

$$m = \frac{dy}{dx} \Big|_{x=a} = \frac{1}{x} \Big|_{x=a} = \frac{1}{a}$$

line
m
pt
(a, ln a)
(x₀, y₀)

$$\boxed{y - y_0 = m(x - x_0)}$$

$$y - \ln a = \left(\frac{1}{a}\right)(x - a) \rightarrow \text{equation of tangent line at } x=a$$

at point (0,0) $\rightarrow x=0, y=0$.

$$0 - \ln a = \left(\frac{1}{a}\right)(0 - a)$$

$$-\ln a = \frac{1}{a}(-a) = -1$$

$$-\ln a = -1$$

$$\ln a = 1$$

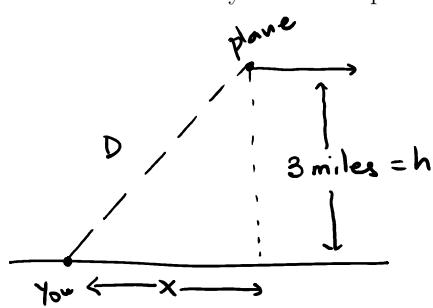
$$a = e^1 = e$$

$$\text{Ans: } \boxed{a=e}$$

$$\frac{dx}{dt} = 500$$

5

Problem 7. A plane is flying directly away from you at 500 mph at an altitude of 3 miles. How fast is the distance between you and the plane increasing at the point when the horizontal distance between you and the plane is 4 miles?



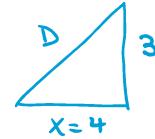
$$x=4$$

$$D^2 = x^2 + h^2$$

$$D^2 = x^2 + (3)^2$$

$$D^2 = x^2 + 9$$

$$\cancel{D \cdot \frac{dD}{dt}} = 2x \frac{dx}{dt} + 0$$



$$\left. \frac{dD}{dt} \right|_{x=4} = \frac{x \frac{dx}{dt}}{D} = \frac{(4)(500)}{5}$$

$$= 400 \text{ mph}$$

$$\frac{dv}{dt}$$

Problem 8. You are inflating a spherical balloon at the rate of $7 \text{ cm}^3/\text{s}$. How fast is the radius of the balloon increasing when the radius is 4 cm?

$$V = \frac{4}{3}\pi r^3$$

$$Q: \left. \frac{dr}{dt} \right|_{r=4}$$

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dV/dt}{4\pi r^2} = \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{7}{4\pi(4)^2} = \frac{7}{64\pi} \text{ cm/s}$$

Problem 9. A company that makes deluxe toasters has a weekly demand equation given by $p(x) = 150e^{-0.02x}$, where $p(x)$ is the price per toaster in dollars, when x toasters are demanded.

- (1) What is the marginal revenue for the company when 60 toasters are sold? Interpret your answer.

$$R(x) = p(x) \cdot x = 150x \cdot e^{-0.02x}$$

$$R'(x) = 150 \cdot (1) \cdot e^{-0.02x} + 150x \left(e^{-0.02x} \cdot \frac{d}{dx} (-0.02x) \right)$$

$$= 150e^{-0.02x} - (0.02)(150x) \cdot e^{-0.02x}$$

$$R'(x) = 150e^{-0.02x} [1 - 0.02x]$$

$$R'(60) = -9.04 \text{ \$/toaster}$$

When 60 toasters are sold, revenue is decreasing
by \$9.04 per toaster sold.

Domain : $[0, \infty)$ \rightarrow ^{x :} talking about toasters \rightarrow must be ≥ 0 .

- (2) At what value of x is the company's revenue increasing? \rightarrow Intervals of increase/decrease.

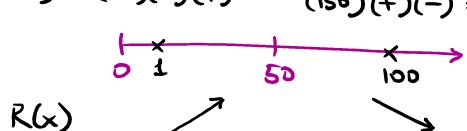
$$R'(x) = 150e^{-0.02x}(1 - 0.02x)$$

find your partitions : $R'(x) = 0$.

$$\underbrace{150e^{-0.02x}}_{\text{never zero}} \underbrace{(1 - 0.02x)}_{1 = 0.02x} = 0 .$$

$$1 = 0.02x \quad \text{or} \quad x = \frac{1}{0.02} = 50$$

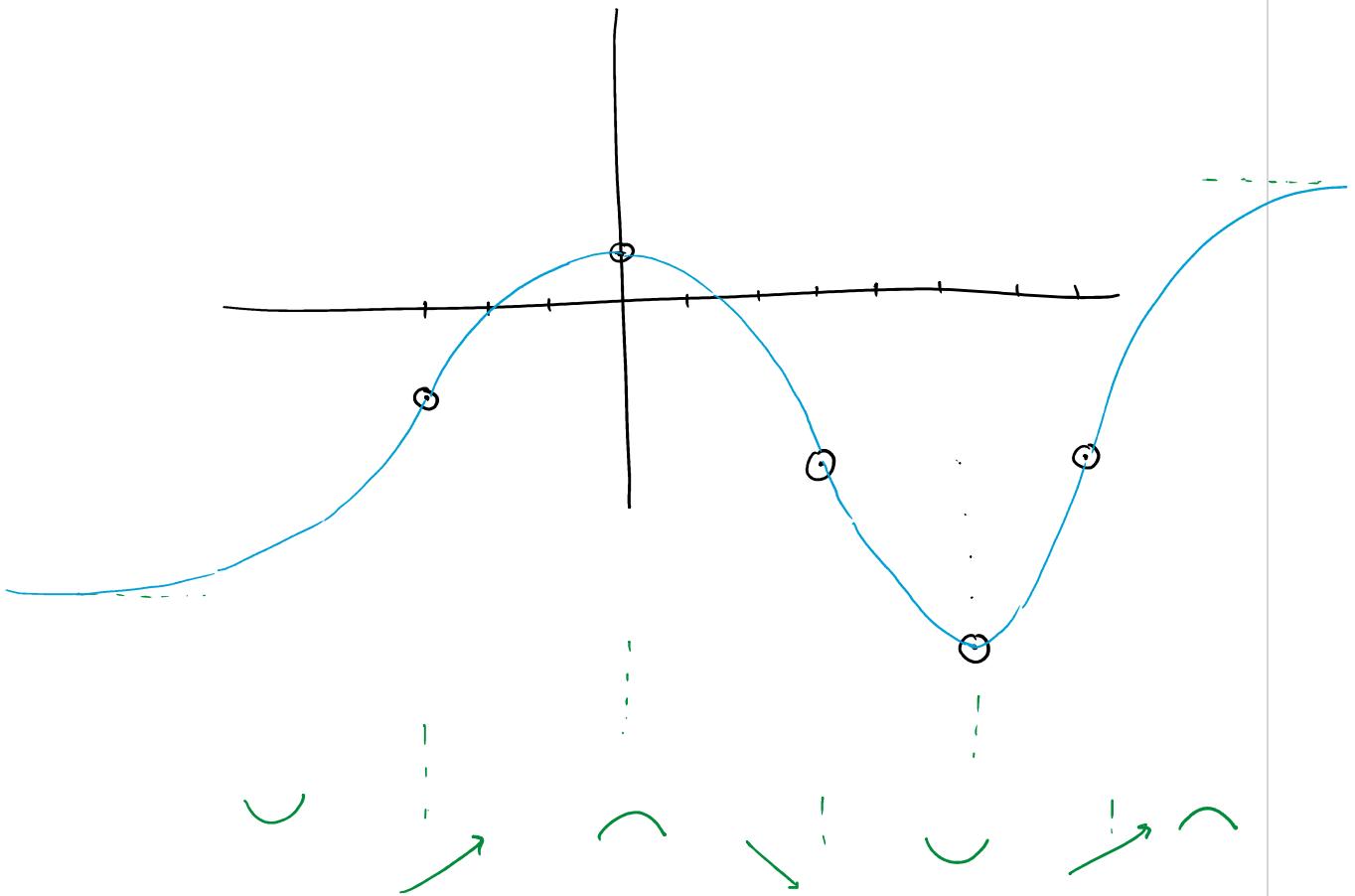
$$R'(x) \quad (150)(+)(+) \quad (150)(+)(-) = (-)$$

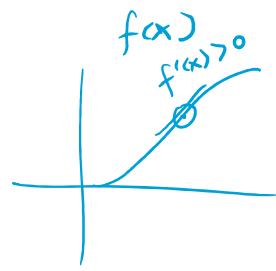


Ans: $R(x)$ is increasing when
toasters sold is $(0, 50)$

Problem 10. Sketch a graph of $f(x)$ given the following conditions:

- a. Domain is $(-\infty, \infty)$
- b. $f(-3) = -2, f(0) = 1, f(3) = -5, f(5) = -7, f(7) = -3$
- c. $\lim_{x \rightarrow \infty} f(x) = 2, \lim_{x \rightarrow -\infty} f(x) = -6$
- d. $f'(x) > 0$ on $(-\infty, 0) \cup (5, \infty)$ $\rightarrow f(x)$ is increasing
- e. $f'(x) < 0$ on $(0, 5)$ $\rightarrow f(x)$ is decreasing
- f. $f''(x) > 0$ on $(-\infty, -3) \cup (3, 7)$ $\rightarrow \cup$
- g. $f''(x) < 0$ on $(-3, 3) \cup (7, \infty)$ $\rightarrow \cap$

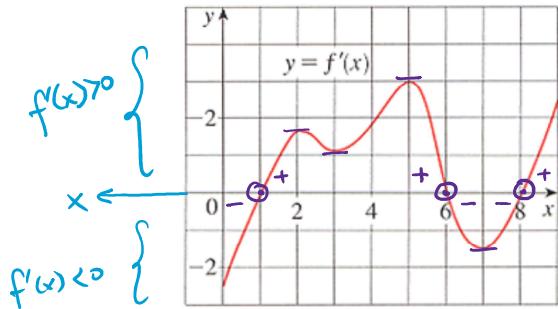




8

$$D: [0, \infty)$$

Problem 11. Given the graph of $f'(x)$ below, find the intervals where $f(x)$



- a) Is increasing $\rightarrow f'(x) > 0$
ie above x-axis
- b) Is decreasing $\rightarrow f'(x) < 0$
- c) Is concave up $f(x): \cup \rightarrow f''(x) > 0 \rightarrow f'$ is increasing
- d) Is concave down $f(x): \cap \rightarrow f''(x) < 0 \rightarrow f'$ is decreasing
- e) Has maxima/minima/inflection pts
 $f'(x) = 0 \quad f''(x) = 0$
 $x = 1, 6, 8$
 $f'(x)$ has max/min
 $x = 2, 3, 5, 7$

a) $(1, 6) \cup (8, \infty)$

b) $(0, 1) \cup (6, 8)$

c) $(0, 2) \cup (3, 5) \cup (7, \infty)$

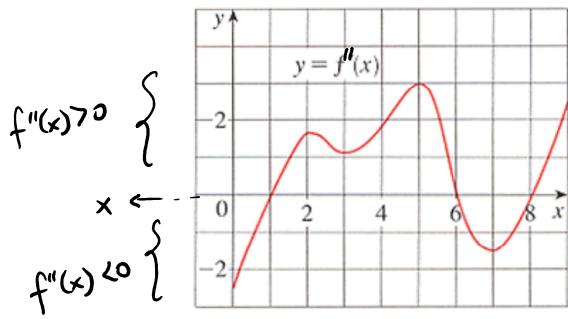
d) $(2, 3) \cup (5, 7)$

e) $f(x)$ has a local maximum $\oplus x=6$

local minima $\ominus x=1, 8$

inflection points $\oplus x=2, 3, 5, 7$

Problem 12. If the given graph is of $f''(x)$ instead, find the intervals where $f(x)$



- a) Is increasing \rightarrow can not say
- b) Is decreasing \rightarrow can not say
- c) Is concave up $\rightarrow f''(x) > 0$
- d) Is concave down $\rightarrow f''(x) < 0$
- e) Has $\underbrace{\text{maxima/minima}}_{\text{can not say}} \text{ and } \underbrace{\text{inflection pts}}_{f''(x)=0}$

- a) DNE
- b) DNE
- c) $(1, 6) \cup (8, \infty)$
- d) $(0, 1) \cup (6, 8)$
- e) $\text{At } x = 1, 6, 8, f(x) \text{ has inflection points}$

$f'(x)$ exists, $f''(x)$ exists.

Problem 13. The domain of $f(x)$ is $(-\infty, \infty)$ and $f(x)$ is twice differentiable on it's domain.

(1) If $f(5) = 10, f'(5) = 0$ and $f''(5) = -1$, what can we say about the function at $x = 5$?

④ $x=5, f(x)$ has a local maxima.

$$\begin{aligned} f'(x) = 0 &\rightarrow \text{CP} \\ f''(5) < 0 &\rightarrow \text{local maximum} \end{aligned} \quad \left. \begin{array}{l} \text{2nd derivative} \\ \text{test} \end{array} \right\}$$

(2) If $f(5) = -2, f'(5) = 0$ and $f''(5) = 3$, what can we say about the function at $x = 5$?

④ $x=5, f(x)$ has a local minima.

(3) If $f(5) = 10, f'(5) = 0$ and $f''(5) = 0$, what can we say about the function at $x = 5$?

2nd derivative test fails.

no information gained

(4) If $f(5) = -2, f'(5) = -1$ and $f''(5) = -2$, what can we say about the function at $x = 5$?

$\neq 0, \therefore \text{not a CP}$

for $f(x)$, $x=1$ is not in Domain of function.

11

Problem 14. For a function $f(x)$, we are given that $f'(x) = \frac{-2x-8}{(x-1)^3}$ and $f''(x) = \frac{2(2x+13)}{(x-1)^4}$. The domain of $f(x)$ is given as $(-\infty, 1) \cup (1, \infty)$.

(1) What are the critical point(s) of the function $f(x)$?

numerator:

$$\begin{aligned}f'(x) &= 0 \\-2x - 8 &= 0 \\-2x &= +8 \\x &= -4\end{aligned}$$

$f'(x)$ DNE
denominator $(x-1) = 0$
 $x=1 \rightarrow$ not in D.
none

(2) Use the Second Derivative Test to determine whether the critical point(s) are local maxima or local minima of $f(x)$?

$$f''(x) = \frac{2(2x+13)}{(x-1)^4}$$

$$f''(-4) = \frac{2(2(-4)+13)}{(-4-1)^4} = \frac{2(-8+13)}{(-5)^4} = \frac{(+)}{(+)} = (+)$$

(3) What are the inflection point(s) of $f(x)$?

$\therefore x=-4$ is a local minimum of $f(x)$.

$\underbrace{f''(x)=0}_{2(2x+13)=0}$. or $f''(x)$ DNE
 $2x = -13$ or $x = -\frac{13}{2}$

(4) What (if any) are the vertical and horizontal asymptotes of $f(x)$?

\downarrow I dont have $f(x)$
I dont have $f(x)$

Problem 15. The price p , in dollars, and the demand x for a product are related by $2x^2 + 5xp + 50p^2 = 80,000$ (Thanks to A.Allen)

- (1) If the price is increasing at a rate of \$2 per month when the price is \$30, find the rate of change of demand with respect to time.

- (2) If the demand is decreasing at a rate of 6 units per month when the demand is 150 units, find the rate of change of price with respect to time.