Math 308: Week-in-Review 8

Shelvean Kapita

1. Express $f(t)$ in terms of the unit step function $u_c(t)$ and find it's Laplace transform

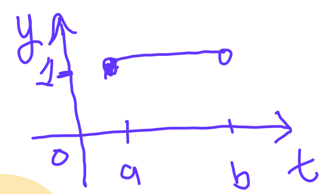
(a)

$$u_c(t) = \begin{cases} 0, & 0 \leq t < c \\ 1, & t \geq c \end{cases}$$

$$f(t) = \begin{cases} (t-2)^2, & 0 \leq t < 2, \\ e^{t-2}, & t \geq 2. \end{cases}$$

$0 \leq a < b$

$$u_a - u_b = \begin{cases} 1 & \text{if } a \leq t < b \\ 0 & \text{otherwise} \end{cases}$$



$$f(t) = (t-2)^2 (u_0 - u_2) + e^{t-2} u_2$$

$$= u_0 (t-2)^2 - u_2 (t-2)^2 + u_2 e^{t-2}$$

(a) (b) (c)

$* \ \mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} \mathcal{L}\{f(t)\}$

$* \ \mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}$

(a) $\mathcal{L}\{u_0 (t-2)^2\} = \mathcal{L}\{t^2 u_0 - 4 t u_0 + 4 u_0\}$

$$= \frac{d^2}{ds^2} \mathcal{L}\{u_0\} + 4 \frac{d}{ds} \mathcal{L}\{u_0\} - 4 \mathcal{L}\{u_0\}$$

$$= \frac{d^2}{ds^2} \frac{1}{s} + 4 \frac{d}{ds} \frac{1}{s} - \frac{4}{s}$$

$$= 2/s^3 - 4/s^2 - 4/s$$

$* \ \mathcal{L}\{t f(t)\} = -\frac{d}{ds} \mathcal{L}\{f(t)\}$

$* \ \mathcal{L}\{t^2 f(t)\} = \frac{d^2}{ds^2} \mathcal{L}\{f(t)\}$

(b) $\mathcal{L}\{u_2 (t-2)^2\} = e^{-2s} \mathcal{L}\{t^2\} = e^{-2s} \cdot 2/s^3$

(c) $\mathcal{L}\{u_2 e^{t-2}\} = e^{-2s} \mathcal{L}\{e^t\} = e^{-2s}/s-1$

$$\mathcal{L}\{f(t)\} = 2/s^3 - 4/s^2 - 4/s - 2e^{-2s}/s^3 + e^{-2s}/s-1$$

(b)

$$f(t) = \begin{cases} 3, & 0 \le t < 2, \\ 2t, & 2 \le t < 4, \\ 3\sin(t-4), & t \ge 4. \end{cases}$$

$$f(t) = 3(u_0 - u_2) + 2t(u_2 - u_4) + u_4 \cdot 3\sin(t-4)$$

$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}$

$$= \underset{(a)}{3u_0} + \underset{(b)}{u_2(2t-3)} - \underset{(c)}{2tu_4} + \underset{(d)}{3u_4 \sin(t-4)}$$

(a) $\mathcal{L}\{3u_0\} = 3/s$

$\mathcal{L}\{u_c(t)\} = \dfrac{e^{-cs}}{s}$

(b) $\mathcal{L}\{u_2(2t-3)\} = \mathcal{L}\{u_2[2(t-2)+1]\}$

$$= 2\mathcal{L}\{u_2(t-2)\} + \mathcal{L}\{u_2\}$$

$$= 2e^{-2s}\mathcal{L}\{t\} + e^{-2s}/s$$

$$= 2e^{-2s}/s^2 + e^{-2s}/s$$

(c) $\mathcal{L}\{2tu_4\} = 2\mathcal{L}\{tu_4\} = 2\mathcal{L}\{u_4[(t-4)+4]\}$

$$= 2\mathcal{L}\{u_4(t-4)\} + 8\mathcal{L}\{u_4\} = 2e^{-4s}\mathcal{L}\{t\} + 8e^{-4s}/s$$

$$= 2e^{-4s}/s^2 + 8e^{-4s}/s$$

(d) $\mathcal{L}\{3u_4 \sin(t-4)\} = 3\mathcal{L}\{u_4 \sin(t-4)\} = 3e^{-4s}\mathcal{L}\{\sin(t)\}$

$\dfrac{1}{s^2+1}$

$$= \frac{3e^{-4s}}{s^2+1}$$

$$\mathcal{L}\{f(t)\} = \frac{3}{s} + \frac{2e^{-2s}}{s^2} + \frac{e^{-2s}}{s} - \frac{2e^{-4s}}{s^2} - \frac{8e^{-4s}}{s} + \frac{3e^{-4s}}{s^2+1}$$

2. Find the inverse Laplace transform of

(a)

$$F(s) = \frac{e^{-5s}}{s(s^4+4)} = \frac{1}{s(s^4+4)}e^{-5s}$$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s(s^4+4)}\right\} = \frac{1}{4}u_5(t) - \frac{1}{8}u_5(t)e^{t-5}\cos(t-5) + \frac{1}{8}u_5(t)e^{-(t-5)}\cos(t-5)}$$

* partial fractions *

use Sympy (apart)

$$\frac{1}{s(s^4+4)} = \frac{1}{4s} - \frac{s-1}{8(s^2-2s+2)} - \frac{s+1}{8(s^2+2s+2)}$$

$$= \frac{1}{4s} - \frac{s-1}{8[(s-1)^2+1]} - \frac{s+1}{8[(s+1)^2+1]}$$

$$\frac{e^{-5s}}{s(s^4+4)} = \frac{e^{-5s}}{4s} - \frac{s-1}{8[(s-1)^2+1]}e^{-5s} - \frac{s+1}{8[(s+1)^2+1]}e^{-5s}$$

$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}$

(e)

* partial fractions *

$$F(s) = \frac{e^{-s} - e^{-4s}}{s(s^2+2s+5)}$$

$$\frac{1}{s(s^2+2s+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+5} = \frac{1}{5s} - \frac{s+2}{5(s^2+2s+5)}$$

$$F(s) = (e^{-s} - e^{-4s})\left\{\frac{1}{5s} - \frac{s+1}{5[(s+1)^2+4]} - \frac{2}{10[(s+1)^2+4]}\right\}$$

$$\boxed{\begin{aligned}\mathcal{L}^{-1}\{F(s)\} &= \frac{1}{5}\left(u_1(t) - u_4(t)\right) - \frac{1}{5}u_1(t)e^{-(t-1)}\left[\cos(2(t-1)) + \frac{1}{2}\sin(2(t-1))\right] \\ &\quad + \frac{1}{5}u_4(t)e^{-(t-4)}\left[\cos(2(t-4)) + \frac{1}{2}\sin(2(t-4))\right]\end{aligned}}$$

3. Solve

(a) $y'' + y = 3\sin(t) \cdot \delta(t - \pi/2)$, $y(0) = 0$, $y'(0) = -1$.

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{3\sin(t) \cdot \delta(t - \pi/2)\}$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + \mathcal{L}\{y\} = 3e^{-\frac{\pi}{2}s}$$

$$\mathcal{L}\{y\}(s^2+1) = 3e^{-\pi/2\, s} - 1$$

$$\mathcal{L}\{y\} = \frac{3e^{-\frac{\pi}{2}s} - 1}{s^2+1} = \frac{3e^{-\frac{\pi}{2}s}}{s^2+1} - \frac{1}{s^2+1}$$

$$y(t) = 3u_{\frac{\pi}{2}}(t)\sin\left(t - \frac{\pi}{2}\right) - \sin(t)$$

OR

$$y(t) = -3u_{\frac{\pi}{2}}(t)\cos(t) - \sin(t)$$

(b) $y'' + 4y' = f(t)$, $y(0) = y'(0) = 0$ where $f(t) = \begin{cases} 1, & 0 \le t < 1, \\ -1, & 1 \le t < 2, \\ 0, & t \ge 2. \end{cases}$

$$f(t) = (u_0 - u_1) - (u_1 - u_2) = u_0 - 2u_1 + u_2$$

$$\mathcal{L}\{y'' + 4y'\} = \mathcal{L}\{u_0 - 2u_1 + u_2\}$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 4s\mathcal{L}\{y\} - 4y(0) = \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s}$$

$$\mathcal{L}\{y\}(s^2 + 4s) = \frac{1}{s}(1 - 2e^{-s} + e^{-2s})$$

$$\mathcal{L}\{y\} = \frac{1 - 2e^{-s} + e^{-2s}}{s(s^2 + 4s)} = \frac{1 - 2e^{-s} + e^{-2s}}{s^2(s+4)}$$

$$= (1 - 2e^{-s} + e^{-2s})\left\{\frac{1}{4s^2} - \frac{1}{16s} + \frac{1}{16(s+4)}\right\}$$

* partial fractions *

$$\frac{1}{s^2(s+4)} = \frac{1}{4s^2} - \frac{1}{16s} + \frac{1}{16(s+4)}$$

$$\boxed{\begin{aligned} y(t) = {} & \left(\frac{1}{4}t - \frac{1}{16} + \frac{1}{16}e^{-4t}\right) - 2u_1(t)\left\{\frac{1}{4}(t-1) - \frac{1}{16} + \frac{1}{16}e^{-4(t-1)}\right\} \\ & + u_2(t)\left\{\frac{1}{4}(t-2) - \frac{1}{16} + \frac{1}{16}e^{-4(t-2)}\right\} \end{aligned}}$$

4. Let $f(t) = e^{-t}$ and $g(t) = \sin(t)$. Compute $(f * g)(t)$ and $(g * f)(t)$. Verify the Convolution Theorem for these functions.

$$(f * g)(t) = \int_0^t f(x)\, g(t-x)\, dx$$

$$(f * g)(t) = \int_0^t e^{-x} \sin(t-x)\, dx = \int_0^t e^{-x} \{\sin(t)\cos(x) - \sin(x)\cos(t)\} dx$$

$$= \sin(t)\int_0^t e^{-x}\cos x\, dx - \cos(t)\int_0^t e^{-x}\sin(x)\, dx$$

$$= \sin(t)\left\{ \frac{e^{-x}\sin x}{2} - \frac{e^{-x}\cos x}{2} \Big|_0^t \right\} - \cos(t)\left\{ \frac{-e^{-x}\cos x}{2} - \frac{e^{-x}\sin x}{2} \Big|_0^t \right\}$$

$$= \frac{e^{-t}}{2}\{\sin^2(t) - \cos(t)\sin(t)\} + \frac{\sin(t)}{2} + \frac{e^{-t}}{2}\{\cos^2(t) + \cos(t)\sin(t)\} - \frac{\cos(t)}{2}$$

$$= \boxed{\frac{e^{-t}}{2} + \frac{1}{2}(\sin(t) - \cos(t))}$$

$$(g * f)(t) = \int_0^t \sin(x)\, e^{-(t-x)}\, dx = e^{-t}\int_0^t e^{x}\sin(x)\, dx$$

$$= e^{-t}\left\{ \frac{e^{x}\sin(x)}{2} - \frac{e^{x}\cos(x)}{2} \Big|_0^t \right\}$$

$$= e^{-t}\left\{ \frac{e^{t}\sin(t)}{2} - \frac{e^{t}\cos(t)}{2} + \frac{1}{2} \right\}$$

$$= \frac{e^{-t}}{2} + \frac{1}{2}(\sin(t) - \cos(t))$$

$$\mathcal{L}\{e^{-t} * \sin t\} = \mathcal{L}\left\{\frac{e^{-t}}{2} + \frac{1}{2}\sin t - \frac{1}{2}\cos t\right\}$$

$$= \frac{1}{2}\frac{1}{(s+1)} + \frac{1}{2(s^2+1)} - \frac{s}{2(s^2+1)}$$

Convolution Theorem

$$\mathcal{L}\{(f*g)(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$$

$$\mathcal{L}\{e^{-t} * \sin t\} = \mathcal{L}\{e^{-t}\}\mathcal{L}\{\sin t\}$$

$$= \frac{1}{(s+1)} \cdot \frac{1}{(s^2+1)}$$

$$= \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

$$1 = A(s^2+1) + (s+1)(Bs+C)$$

$s=-1$: $A = 1/2$

$s=0$: $1 = 1/2 + C \Rightarrow C = 1/2$

$s=1$: $1 = 1 + 2(B + 1/2) \Rightarrow B = -1/2$

$$\mathcal{L}\{e^{-t} * \sin t\} = \frac{1}{2(s+1)} + \frac{1}{2(s^2+1)} - \frac{s}{2(s^2+1)}$$

5. (a) Find the Laplace transform of $h(t) = \int_0^t e^{t-x} \sin(x)\, dx$

$$h(t) = e^t * \sin(t)$$

$$H(s) = \mathcal{L}\{h(t)\} = \mathcal{L}\{e^t * \sin t\}$$

$$= \mathcal{L}\{e^t\}\mathcal{L}\{\sin t\}$$

$$= \frac{1}{s-1} \cdot \frac{1}{s^2+1}$$

$$\mathcal{L}\{h(t)\} = \frac{1}{(s+1)(s^2+1)}$$

(b) Find the inverse Laplace transform using the Convolution Theorem

$$F(s) = \frac{1}{(s^2+9)(s-2)}$$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= \frac{1}{3}\sin(3t) * e^{2t}$$

$$= \frac{1}{3}\int_0^t e^{2(t-x)} \sin(3x)\, dx$$

6. Use Laplace transforms to solve the integro-differential equation

$$y' - 4y + 4\int_0^t y(x)\,dx = t^3 e^{2t} \quad , y(0) = 0$$

$$\mathcal{L}\{y' - 4y + 4\, 1 * y\} = \mathcal{L}\{t^3 e^{2t}\}$$

$$s\mathcal{L}\{y\} - \cancel{y(0)} - 4\mathcal{L}\{y\} + 4\mathcal{L}\{1\}\mathcal{L}\{y\} = \frac{6}{(s-2)^4}$$

$$\mathcal{L}\{y\}\left(s - 4 + 4/s\right) = \frac{6}{(s-2)^4}$$

$$\mathcal{L}\{y\}\left(\frac{s^2-4s+4}{s}\right) = \frac{6}{(s-2)^4}$$

$$\mathcal{L}\{y\} = \frac{6s}{(s-2)^6} = \frac{6}{(s-2)^5} + \frac{12}{(s-2)^6}$$

$$= \frac{4!}{4(s-2)^5} + \frac{5!}{10(s-2)^6}$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$y(t) = \frac{1}{4}t^4 e^{2t} + \frac{1}{10}t^5 e^{2t}$$