Math 308: Week-in-Review 10

Shelvean Kapita

Review for Exam 2

1. (3.5, 3.6) Find the general solution of the second order differential equation

   (a) $y'' + 4y' + 5y = e^{-2t}\sin t$

* second order linear, nonhomogeneous constant coefficients

$$y(t) = y_c(t) + y_p(t)$$

homogeneous ↳ particular

* find homogeneous solution *

$$r^2 + 4r + 5 = 0 \Rightarrow r = \frac{-4 \pm \sqrt{4^2 - 4\cdot 5}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$$

$$y_c(t) = c_1 e^{-2t}\cos(t) + c_2 e^{-2t}\sin(t)$$

$$y_1(t) = e^{-2t}\cos(t), \quad y_2(t) = e^{-2t}\sin(t)$$

* Method of variation of parameters *

$$y_p = u_1 y_1 + u_2 y_2 \quad \text{where} \quad u_1(t) = \int -\frac{y_2(t)\, r(t)}{W[y_1,y_2](t)}\,dt \quad \& \quad u_2(t) = \int \frac{y_1(t)\, r(t)}{W[y_1,y_2](t)}\,dt$$

$$W[y_1,y_2](t) = \begin{vmatrix} e^{-2t}\cos t & e^{-2t}\sin t \\ -2e^{-2t}\cos t - e^{-2t}\sin t & -2e^{-2t}\sin t + e^{-2t}\cos t \end{vmatrix} = e^{-4t}\cos^2 t - \cancel{2e^{-4t}\cos t\sin t} + \cancel{2e^{-4t}\cos t\sin t} + e^{-4t}\sin^2 t = e^{-4t}$$

$r(t) = e^{-2t}\sin t$ (right hand side) ← standard form!

$$\cos 2t = \cos^2 t - \sin^2 t = 1 - 2\sin^2 t = 2\cos^2 t - 1$$

$$\sin^2 t = \frac{1}{2}(1 - \cos 2t)$$

$$u_1 = \int -\frac{e^{-2t}\sin t \cdot e^{-2t}\sin t}{e^{-4t}}\,dt = \int -\sin^2 t\,dt = \frac{1}{2}\int(\cos 2t - 1)\,dt$$

$$u_1 = \frac{1}{4}\sin 2t - \frac{t}{2}$$

$$u_2 = \int \frac{e^{-2t}\cos t \cdot e^{-2t}\sin t}{e^{-4t}}\,dt = \int \cos t\cdot\sin t\,dt = \frac{1}{2}\int \sin(2t)\,dt = -\frac{1}{4}\cos(2t)$$

$$y_p(t) = u_1y_1 + u_2y_2 = \frac{1}{4}e^{-2t}\cos t\sin(2t) - \frac{t}{2}e^{-2t}\cos t - \frac{1}{4}e^{-2t}\sin t\cos(2t)$$

homogeneous

$$= \frac{1}{2}e^{-2t}\cancel{\cos^2 t\sin t} - \frac{t}{2}e^{-2t}\cos t - \frac{1}{2}e^{-2t}\cancel{\cos^2 t\sin t} - \frac{1}{4}e^{-2t}\sin t$$

$$y_p = -\frac{t}{2}e^{-2t}\cos t$$

general solution of the non-homogeneous eqn.

$$y(t) = c_1e^{-2t}\cos t + c_2e^{-2t}\sin t - \frac{t}{2}e^{-2t}\cos t$$

(b) (3.6) The general solution of the homogeneous equation $x^2y'' - 3xy' + 4y = 0$, $x > 0$, is given by $y_c(x) = c_1x^2 + c_2x^2 \ln x$. Find the general solution of the nonhomogeneous equation $x^2y'' - 3xy' + 4y = x^2 \ln x$, $x > 0$.

* method of variation of parameters *

$y(x) = y_c(x) + y_p(x) = c_1x^2 + c_2x^2 \ln x + y_p(x)$, where $y_p(x) = u_1y_1 + u_2y_2$

$y_1(x) = x^2$, $y_2(x) = x^2 \ln x$, $u_1 = \int \frac{-y_2(x)r(x)}{W[y_1,y_2]}dx$, $u_2 = \int \frac{y_1(x)r(x)}{W[y_1,y_2]}dx$

$r(x) = \frac{x^2 \ln x}{x^2} = \ln x$ (right hand side) ← in standard form! $y'' + p(x)y' + q(x)y = r(x)$

$$W[y_1,y_2] = \begin{vmatrix} x^2 & x^2\ln x \\ 2x & 2x\ln x + x \end{vmatrix} = \cancel{2x^3\ln x} + x^3 - \cancel{2x^3 \ln x} = x^3$$

$u = \ln x \Rightarrow du = \frac{1}{x}dx$

$$u_1 = \int \frac{-x^2\ln x \cdot \ln x}{x^3}dx = \int -\frac{(\ln x)^2}{x}dx = \int -u^2du = -\frac{u^3}{3} = -\frac{(\ln x)^3}{3}$$

$$u_2 = \int \frac{x^2 \ln x}{x^3}dx = \int \frac{\ln x}{x}dx = \int u\,du = \frac{u^2}{2} = \frac{(\ln x)^2}{2}$$

$$y_p(x) = -\frac{x^2(\ln x)^3}{3} + \frac{x^2(\ln x)^3}{2} = \frac{x^2(\ln x)^3}{6}$$

$$y(x) = \underbrace{c_1x^2 + c_2x^2\ln x}_{\text{homogeneous}} + \underbrace{\frac{x^2(\ln x)^3}{6}}_{\text{particular}}$$

2. (3.7, 3.8) A string is stretched 10 cm by a force of 0.3 N. A mass of 0.25 kg is hung from the spring, and also attached to a viscous damper that exerts a force of 3 N when the velocity of the mass is 6 m/s. The mass is pulled down 5 cm below its equilibrium position and given an initial velocity of 10 cm/s downward.

   (a) Determine the position u as a function of time t

   (b) Find the quasifrequency of the motion.

   (c) If this system is also subjected to an external force $F(t) = 2\cos(4t)$, find $u(t)$, and the amplitude, period, and phase of the steady state motion.

$mu'' + cu' + ku = F_{ext}$

F_{ext} = external force $= 0$

* m = mass $= 0.25$ kg $= \frac{1}{4}$ kg
* c = damping coeff $= \frac{F}{v} = \frac{3N}{6\,m/s} = \frac{1}{2}$ Ns/m
* k = spring const $= \frac{F}{\Delta u} = \frac{0.3N}{0.1m} = 3$ N/m

$$\begin{cases} \frac{1}{4}u'' + \frac{1}{2}u' + 3u = 0 \\ u(0) = 0.05m,\ u'(0) = 0.1m/s \end{cases} \Leftrightarrow u'' + 2u' + 12u = 0$$

(a) $\lambda^2 + 2\lambda + 12 = 0 \Rightarrow \lambda = \frac{-2 \pm \sqrt{2^2 - 4\cdot 12}}{2} = \frac{-2 \pm \sqrt{-44}}{2} = -1 \pm \sqrt{11}$

$u(t) = c_1 e^{-t}\cos(\sqrt{11}\,t) + c_2 e^{-t}\sin(\sqrt{11}\,t)$, $u(0) = c_1 = 0.05m$

$u'(t) = -c_1 e^{-t}\cos(\sqrt{11}t) - \sqrt{11}c_1 e^{-t}\sin(\sqrt{11}t) - c_2 e^{-t}\sin(\sqrt{11}t) + \sqrt{11}c_2 e^{-t}\cos(\sqrt{11}t)$

$u'(0) = -c_1 + \sqrt{11}c_2 = -0.05 + \sqrt{11}c_2 = 0.1 \Rightarrow \sqrt{11}c_2 = 0.15$

$c_2 = \frac{0.15}{\sqrt{11}}$

$$u(t) = 0.05e^{-t}\cos(\sqrt{11}t) + \frac{0.15}{\sqrt{11}}e^{-t}\sin(\sqrt{11}t)$$

(b) Quasifrequency

$\sqrt{11}$ radians/second

(c) $\frac{1}{4}u'' + \frac{1}{2}u' + 3u = 2\cos(4t) \Rightarrow u'' + 2u' + 12u = 8\cos(4t)$

$u = u_c + u_p$ → steady state

u_c: tends to zero as $t \to \infty$ (transient)

$u_p(t) = A\cos(4t) + B\sin(4t)$

$u_p'(t) = -4A\sin(4t) + 4B\cos(4t)$

$u_p''(t) = -16A\cos(4t) - 16B\sin(4t)$

$R = \sqrt{A^2+B^2}$ → amplitude

$\tan\delta = \frac{B}{A}$ → phase

$$u_p'' + 2u_p' + 12u_p = 8\cos(2t)$$

$\cos(2t)$: $-16A + 8B + 12A = 8$ $\quad$ $8B - 4A = 8$

$\sin(2t)$; $-16B - 8A + 12B = 0$ $\quad$ $-4B - 8A = 0$

$B = -2A$ $\quad$ $A = -2/5$, $B = 4/5$

$$u_p(t) = -2/5 \cos(4t) + 4/5 \sin(4t)$$

Amplitude : $R = \sqrt{A^2+B^2} = \sqrt{(-2/5)^2+(4/5)^2} = \frac{\sqrt{20}}{5} = \frac{2\sqrt{5}}{5} = \frac{2}{\sqrt{5}}$

Phase ; $\tan\varphi = B/A = \frac{4/5}{-2/5} = -2$

$$\varphi = \arctan(-2) + \pi = \pi - \arctan(2) \quad (\text{rad})$$

$\omega = 4$ rad/sec , $T = \frac{2\pi}{\omega} = \frac{\pi}{2}$ sec

$$u(t) = u_c(t) + u_p(t) = e^{-t}\left(c_1\cos(\sqrt{11}\,t) + c_2\sin(\sqrt{11}\,t)\right)$$

transient solution

$+$

$$-2/5\cos(4t) + 4/5\sin(4t)$$

steady - state

3. (6.1) Find the Laplace transform of the following function using the definition of Laplace transform

(a) $f(t) = \begin{cases} t, & 0 \leq t < 1, \\ 2-t, & 1 \leq t < 2, \\ 0, & t \geq 0 \end{cases}$

* definition *

$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$\mathcal{L}\{f(t)\} = \int_0^1 e^{-st}\, t\, dt + \int_1^2 2e^{-st}\,dt - \int_1^2 t e^{-st}\,dt$

$$\begin{array}{c|c|c} & t & e^{-st} \\ \hline + & 1 & -\frac{1}{s}e^{-st} \\ \hline - & 0 & \frac{1}{s^2}e^{-st} \end{array}$$

$= -\frac{t}{s}e^{-st}\Big|_0^1 - \frac{1}{s^2}e^{-st}\Big|_0^1 - \frac{2}{s}e^{-st}\Big|_1^2 + \frac{t}{s}e^{-st}\Big|_1^2 + \frac{1}{s^2}e^{-st}\Big|_1^2$

$= -\cancel{\frac{1}{s}e^{-s}} - \frac{1}{s^2}e^{-s} + \frac{1}{s^2} - \cancel{\frac{2}{s}e^{-2s}} + \cancel{\frac{2}{s}e^{-s}} + \cancel{\frac{2}{s}e^{-2s}} - \cancel{\frac{1}{s}e^{-s}} + \frac{1}{s^2}e^{-2s} - \frac{1}{s^2}e^{-s}$

$= -\frac{2}{s^2}e^{-s} + \frac{1}{s^2} + \frac{1}{s^2}e^{-2s}$

(b) Find the Laplace transform of the above function using Heaviside unit step functions.

$f(t) = t(u_0 - u_1) + (2-t)(u_1 - u_2) = tu_0 - tu_1 + 2u_1 - tu_1 + u_2(t-2)$

$= tu_0 - 2u_1(t-1) + u_2(t-2)$

* $\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}$ *

$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}$

$\mathcal{L}\{f(t)\} = \frac{1}{s^2} - \frac{2e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}$

4. (6.2, 6.3) Find the inverse Laplace transform of the function

$$F(s) = \frac{s^2+s+1}{(s^2+4)(s^2-9)} = \frac{s^2+s+1}{(s^2+4)(s+3)(s-3)}$$

$$= \frac{A}{s+3} + \frac{B}{s-3} + \frac{Cs+D}{s^2+4}$$

* partial fractions decomposition *

$$s^2+s+1 = A(s-3)(s^2+4) + B(s+3)(s^2+4) + (Cs+D)(s+3)(s-3)$$

$s=3$: $13 = B(6)(13) \Rightarrow B = 1/6$

$s=-3$: $7 = A(-6)(13) \Rightarrow A = -\frac{7}{78}$

$s=2i$: $-4+2i+1 = -3+2i = (2iC+D)(-4-9) \Rightarrow -3+2i = -26iC - 13D$

* real component: $-3 = -13D \Rightarrow D = 3/13$

* imaginary component: $2i = -26iC \Rightarrow C = -\frac{1}{13}$

$$F(s) = -\frac{7}{78(s+3)} + \frac{1}{6(s-3)} - \frac{s-3}{13(s^2+4)}$$

$$\mathcal{L}^{-1}\left\{\frac{-7}{78(s+3)}\right\} = \frac{-7}{78}e^{-3t}, \quad \mathcal{L}^{-1}\left\{\frac{1}{6(s-3)}\right\} = \frac{1}{6}e^{3t}$$

$$\frac{s-3}{13(s^2+4)} = \frac{s}{13(s^2+4)} - \frac{3}{13}\frac{2}{(s^2+4)}\cdot\frac{1}{2} = \frac{s}{13(s^2+4)} - \frac{3}{26}\frac{2}{s^2+4}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{13(s^2+4)}\right\} = \frac{1}{13}\cos(2t), \quad \mathcal{L}^{-1}\left\{\frac{3}{26}\frac{2}{s^2+4}\right\} = \frac{3}{26}\sin(2t)$$

$$f(t) = -\frac{7}{78}e^{-3t} + \frac{1}{6}e^{3t} - \frac{1}{13}\cos(2t) + \frac{3}{26}\sin(2t)$$

5. (6.2, 6.3) Find the inverse Laplace transform of the function

$$F(s) = \frac{e^{-2s}(s^2+s+1)}{s(s+2)^2}$$

$$\frac{s^2+s+1}{s(s+2)^2} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

$$s^2+s+1 = A(s+2)^2 + Bs(s+2) + Cs$$

$s=0$: $1 = 4A \Rightarrow A = 1/4$

$s=-2$: $3 = -2C \Rightarrow C = -3/2$

$s=1$: $3 = \frac{9}{4} + 3B - 3/2 \Rightarrow 3B = \frac{12}{4} - \frac{9}{4} + \frac{6}{4}$
$= \frac{9}{4}$

$B = 3/4$

$$= e^{-2s}\left[\frac{1}{4s} + \frac{3}{4(s+2)} - \frac{3}{2(s+2)^2}\right]$$

$$= \frac{e^{-2s}}{4s} + \frac{3e^{-2s}}{4(s+2)} - \frac{3e^{-2s}}{2(s+2)^2}$$

$* \ \mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} \ *$

$$f(t) = \frac{1}{4}u_2(t) + \frac{3}{4}u_2(t)e^{-2(t-2)} - \frac{3}{2}u_2(t)e^{-(t-2)}[t-2]$$

6. (6.3, 6.4) Find the solution of the initial value problem

(a) $y'' + 2y' + y = \begin{cases} \sin 2(t - \pi/2), & 0 \le t < \pi/2, \\ 0, & \pi/2 \le t < \infty \end{cases}, \quad y(0) = 1, \quad y'(0) = 0.$

$$f(t) = \sin\left[2\left(t - \tfrac{\pi}{2}\right)\right]\left[u_0 - u_{\pi/2}\right] = \sin\left(2\left(t - \tfrac{\pi}{2}\right)\right) - u_{\frac{\pi}{2}}(t)\sin\left[2\left(t - \tfrac{\pi}{2}\right)\right]$$
$$= \sin\left[2t - \pi\right] - u_{\frac{\pi}{2}}(t)\sin\left[2\left(t - \tfrac{\pi}{2}\right)\right]$$

$$s^2\mathcal{L}\{y\} - sy(0) - \cancel{y'(0)} + 2s\mathcal{L}\{y\} - 2y(0) + \mathcal{L}\{y\} = -\frac{2}{s^2+4} - \frac{2e^{-\pi/2\, s}}{s^2+4}$$

$$\mathcal{L}\{y\}\left(s^2 + 2s + 1\right) = s + 2 - \frac{2}{s^2+4} - \frac{2e^{-\frac{\pi}{2}s}}{s^2+4}$$

$$\mathcal{L}\{y\} = \frac{s+2}{(s+1)^2} - \frac{2}{(s+1)^2(s^2+4)} - \frac{2e^{-\frac{\pi}{2}s}}{(s+1)^2(s^2+4)}$$

$$\frac{s+2}{(s+1)^2} = \frac{s+1+1}{(s+1)^2} = \frac{1}{s+1} + \frac{1}{(s+1)^2} \xmapsto{\mathcal{L}^{-1}} e^{-t} + te^{-t}$$

$$\frac{-2}{(s+1)^2(s^2+4)} = -\frac{1}{(s+1)^2} \cdot \frac{2}{s^2+4} \xmapsto{\mathcal{L}^{-1}} -\left(te^{-t}\right) * \sin(2t)$$

$$\frac{-2e^{-\frac{\pi}{2}s}}{(s+1)^2(s^2+4)} = -\frac{1}{(s+1)^2} \cdot \frac{2e^{-\frac{\pi}{2}s}}{s^2+4} \xmapsto{\mathcal{L}^{-1}} -\left(te^{-t}\right) * u_{\frac{\pi}{2}}(t)\sin\left[2\left(t - \tfrac{\pi}{2}\right)\right]$$
$$= \left(te^{-t}\right) * u_{\pi/2}(t)\sin(2t)$$

$$y(t) = e^{-t} + te^{-t} - \int_0^t xe^{-x}\sin(2(t-x))\,dx + \int_0^t xe^{-x} \cdot u_{\pi/2}(t-x)\sin(2(t-x))\,dx$$

(b) (6.5) $y'' + 2y' + y = e^{-t} + \delta(t-3),\ y(0) = 0,\ y'(0) = 3.$

$$s^2\mathcal{L}\{y\} - s\cancel{y(0)} - y'(0) + 2s\mathcal{L}\{y\} - 2\cancel{y(0)} + Y(s) = \frac{1}{s+1} + e^{-3s}$$

$$\mathcal{L}\{y\}(s^2+2s+1) = 3 + \frac{1}{s+1} + e^{-3s}$$

$$\mathcal{L}\{y\} = \frac{3}{(s+1)^2} + \frac{1}{(s+1)^3} + \frac{e^{-3s}}{(s+1)^2}$$

$$y(t) = 3te^{-t} + \frac{1}{2}t^2e^{-t} + u_3(t)e^{-(t-3)}[t-3]$$

where we used $* \ \mathcal{L}\{e^{at}f(t)\} = F(s-a) \ *$

and $* \ \mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} \ *$

7. (6.6)

(a) Use the definition of convolution to compute $(t * \sin t)$.

$$(t * \sin t) = \int_0^t (t-x)\sin(x)\,dx$$
$$= \int_0^t t\sin(x)\,dx - \int_0^t x\sin x\,dx$$
$$= -t\cos(x)\Big|_0^t + x\cos x\Big|_0^t - \sin x\Big|_0^t$$
$$= -t\cancel{\cos} t + t + t\cancel{\cos} t - \sin t$$
$$= t - \sin t$$

$$\begin{array}{c|c} x & \sin x \\ + \quad 1 & -\cos x \\ - \quad 0 & -\sin x \end{array}$$

(b) Use the Convolution Theorem to find the inverse Laplace transform of

$$F(s) = \frac{s}{(s+1)(s^2+4)}$$

$$F(s) = \frac{1}{s+1} \cdot \frac{s}{s^2+4}$$
$$= \mathcal{L}\{e^{-t}\}\mathcal{L}\{\cos(2t)\}$$

$$\mathcal{L}^{-1}\{F(s)\} = e^{-t} * \cos(2t) = \int_0^t e^{-(t-x)}\cos(2x)\,dx$$
$$= e^{-t}\int_0^t e^{x}\cos(2x)\,dx$$
$$= e^{-t}\left[\frac{2}{5}e^{x}\sin(2x) + \frac{e^{x}\cos(2x)}{5}\right]_0^t$$
$$= e^{-t}\left[\frac{2}{5}e^{t}\sin(2t) + \frac{1}{5}e^{t}\cos(2t) - \frac{1}{5}\right]$$
$$= \frac{2}{5}\sin(2t) + \frac{1}{5}\cos(2t) - \frac{1}{5}e^{-t}$$

8. Find the radius and interval of convergence of the series

$$\sum_{n=0}^{\infty}(-1)^n \frac{2^n n!}{3 \cdot 5 \cdots (2n+1)} x^{2n+1}$$

* by the ratio test *

$$\left| \frac{(-1)^{n+1} 2^{n+1} (n+1)! \, x^{2n+3}}{3 \cdot 5 \cdots (2n+3)} \cdot \frac{3 \cdot 5 \cdots (2n+1)}{(-1)^n 2^n n! \, x^{2n+1}} \right|$$

$$= |x^2| \lim_{n \to \infty} \frac{2(n+1)}{(2n+3)}$$

$$= |x^2| < 1 \Rightarrow |x| < 1$$

* radius of convergence * $R = 1$

* interval of convergence * $-1 < x < 1$

9. (5.2) Consider the initial value value problem

$$y'' + x^2 y' + 2xy = 0, \;\; y(0) = 1, \;\; y'(0) = 0.$$

(a) Solve the initial value problem using a series of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Find the recurrence relation.

(b) Find the first 6 terms of the series solution.

(c) Write down the solution using summation notation.

(a) $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^{n+1} + \sum_{n=0}^{\infty} 2 a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1} (n-1) a_{n-1} x^n + \sum_{n=1} 2 a_{n-1} x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} + (n-1) a_{n-1} + 2 a_{n-1} \right] x^n = 0$$

$a_2 = 0$, $a_0 = 1$, $a_1 = 0$

$$a_{n+2} = -\frac{2 + (n-1)}{(n+2)(n+1)} a_{n-1} = \frac{-1}{n+2} a_{n-1}, \quad n \geq 1$$

$a_3 = -\frac{1}{3} a_0 = -\frac{1}{3}$, $a_4 = 0$, $a_5 = 0$, $a_6 = -\frac{1}{6} a_3 = \frac{1}{3 \cdot 6}$

$a_7 = 0$, $a_8 = 0$, $a_9 = -\frac{1}{9} a_6 = -\frac{1}{3 \cdot 6 \cdot 9}$

$= -\frac{1}{3 \cdot 6 \cdot 9}$; $a_{3n} = (-1)^n \frac{1}{3 \cdot 6 \cdot 9 \cdots (3n)}$

(b) $y(x) = 1 - \frac{1}{3} x^3 + \frac{1}{3 \cdot 6} x^6 - \frac{1}{3 \cdot 6 \cdot 9} x^9 + \frac{1}{3 \cdot 6 \cdot 9 \cdot 12} x^{12} - \cdots$

(c) $= \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n}}{3 \cdot 6 \cdots (3n)} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n}}{3^n n!} = \sum_{n=0}^{\infty} \left(-\frac{x^3}{3} \right)^n \frac{1}{n!} = e^{-x^3/3}$