Math 150 - Week-In-Review 8

Sana Kazemi

Problem Statements

1. Solve each of the following. **Always check for extraneous solutions.**

(a) $\log_5(4y) = 3$

check for extraneous solutions: $4y > 0 \to y > 0 \Rightarrow$ domain $y \in (0, \infty)$

$\log_5(4y) = 3 \iff 4y = 5^3 \to \boxed{y = \frac{125}{4}}$ ✓

(b) $\log_9(x+2) = \log_{27}(6)$

different base ⇝ take common base first

check for extraneous solutions:

$x+2 > 0 \longrightarrow x > -2 \Rightarrow$ domain $x \in (-2, \infty)$

$9 = 3^2$ & $27 = 3^3$

$\frac{\log_3(x+2)}{\log_3 9} = \frac{\log_3 6}{\log_3 27} \to \frac{\log_3(x+2)}{2} = \frac{\log_3 6}{3} \to \frac{1}{2}\log_3(x+2) = \frac{1}{3}\log_3 6$

$\log_3(x+2)^{\frac{1}{2}} = \log_3(6)^{\frac{1}{3}} \Rightarrow \sqrt{x+2} = (6)^{\frac{1}{3}} \xrightarrow[\text{both sides}]{\text{square}} (x+2) = \underbrace{6^{2/3}}_{\sqrt[3]{6^2}}$

$\Rightarrow \boxed{x = \sqrt[3]{36} - 2}$ in domain ✓

(c) $\log_5(x) + \log_5(x+4) = 1$

$x > 0$ & $x + 4 > 0$

$x > -4$

domain: $x \in (0, \infty)$

$\log_5 x(x+4) = 1 \iff x^2 + 4x = 5^1 \implies x^2 + 4x - 5 = 0$

$(x+5)(x-1) = 0$

$x = -5$ ✗ not in domain

& $\boxed{x = 1}$ ✓

2. find domain of the function $f(x) = \dfrac{\sqrt{-x+5} + e^{3x}}{\log_3(x+2)}$

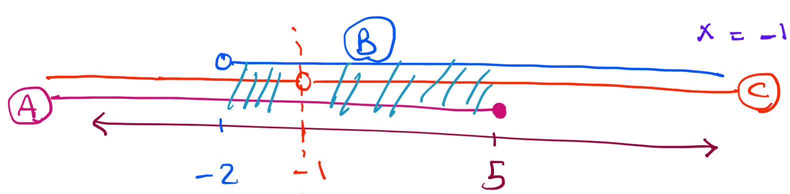
Domain restrictions: $\sqrt{5-x}$: $5 - x \geq 0 \rightarrow x \leq 5$ (A)

$\log_3(x+2)$: $x + 2 > 0 \rightarrow x > -2$ (B)

& From denominator $\log_3(x+2) \neq 0$ $\log_3(x+2) = 0 \iff (x+2) = \underbrace{3^0}_{=1}$

$x + 2 = 1$

$x = -1$ so $x \neq -1$ (C)



$x \in (-2, -1) \cup (-1, 5]$

3. If an investment of \$2000 grows to \$2500 after 3 years with an annual interest rate of 4%, compounded annually, find the time it takes for an investment of \$2000 to grow to \$3000.

$n=1$ P r $t=?$ $A(t)$

$$A(t) = P\left(1+\frac{r}{n}\right)^{nt}$$

P: initial principal, r: annual interest rate

n: # of compounds per year t: # of years

A(t): final amount

$$3000 = 2000\left(1+\frac{0.04}{1}\right)^{1(t)} \rightarrow 3 = 2(1+0.04)^t$$

$$\frac{3}{2} = (1.04)^t \rightarrow \ln(3/2) = t\ln(1.04)$$

$$t = \frac{\ln(3/2)}{\ln(1.04)} \approx 10.32 \text{ years}$$

4. If you invest \$2000 in an account with an annual interest rate of 4%, compounded continuously, how much money will you have after 8 years?

P r t $A = P \cdot e^{rt}$

$$A = 2000\left(e^{0.04\times 8}\right) = 2000 \cdot e^{0.32} \approx \$ 298.20$$

5. If the amount of a radioactive substance decreases to one-third of its initial amount in 20 years, find the half-life of the substance.

$\underbrace{\text{find the half-life of the substance.}}$ time it takes for half of the substance to decay.

radioactive Decay: $A(t) = A_0 e^{kt}$

A_0 initial amt, $k<0$ constant of porportionality

$$\frac{A_0}{3} = A_0 e^{k(20)} \to e^{k(20)} = \frac{1}{3} \to k = \frac{\ln(\frac{1}{3})}{20} = \frac{1}{20}\ln(\frac{1}{3})$$

$$\frac{A_0}{2} = A_0 e^{kt} \rightsquigarrow e^{(\frac{1}{20}\ln(\frac{1}{3}))t} = \frac{1}{2} \to \frac{1}{20}\cdot\ln(\frac{1}{3})\cdot t = \ln(\frac{1}{2})$$

$$t = \frac{20\ln(\frac{1}{2})}{\ln(\frac{1}{3})} \text{ years}$$

6. The number of bacteria y in a culture after t days is given by the function $y(t) = 100e^{t/8}$.After how many days will there be 4,000 bacteria?

$$4000 = 100e^{\frac{t}{8}} \to 40 = e^{t/8}$$

$$\ln(40) = \frac{t}{8} \to t = 8\ln(40)$$

7. A cup of coffee cools from 80°C to 70°C in 5 minutes. If the room temperature is 25°C, what will be the temperature of the coffee after 15 minutes?

Newton's Law of Cooling $T(t) = T_a + (T_o - T_a)e^{-kt}$

T_o : initial temp. T_a : surrounding temp. $k>0$ Constant of porp.

Find k: $70 = 25 + (80-25)e^{-k(5)} \rightsquigarrow 45 = (55)e^{-k(5)}$

$\frac{45}{55} = e^{-5k} \to \ln(\frac{9}{11}) = -5k \to k = \frac{\ln(9/11)}{-5}$

$T(15) = 25 + (80-25)e^{+\frac{\ln(9/11)}{5}\cdot 15} = 25 + (55)e^{+3\ln(9/11)}$

$= 25 + 55e^{\ln(9/11)^{+3}} \to T(15) = 25 + 55(\frac{9}{11})^{+3}$

8. A population of rabbits can be modeled using the logistic equation

$$N(t) = \frac{1000}{1 \cancel{+0.6}\,24e^{-0.18t}}$$

How long does it take for population of rabbits to grow to 4200?

$$42\cancel{00} = \frac{10\cancel{00}}{1 - 24e^{-0.18t}} \xrightarrow[\text{multiply}]{\text{Cross}} 42(1 - 24e^{-0.18t}) = 10$$

$$1 - 24e^{-0.18t} = \frac{\cancel{10}^{5}}{\cancel{42}_{21}} \rightarrow -24e^{-0.18t} = \frac{5}{21} - 1$$

$$-24e^{-0.18t} = \frac{-16}{21}$$

$$e^{-0.18t} = \frac{16}{21 \times 24} \rightarrow -0.18t = \ln\left(\frac{16}{21 \times 24}\right)$$

$$t = \frac{\ln\left(\frac{\cancel{16}^{2}}{21 \times \cancel{24}_{3}}\right)}{-0.18} = \frac{\ln\left(\frac{2}{63}\right)}{-0.18}$$

9. Perform the operation $\frac{x^2 + 5x - 14}{x^2 + 8x + 7} \div \frac{x^2 - x - 2}{x - 3}$ and simplify.

$$= \frac{\cancel{(x+7)}\cancel{(x-2)}}{\cancel{(x+7)}(x+1)} \cdot \frac{x-3}{\cancel{(x-2)}(x+1)}$$

$$= \frac{x-3}{(x+1)^2}$$

10. For the function $g(x) = \sqrt{6-2x}$ compute and simplify the difference quotient.

$$g(x+h) = \sqrt{6-2(x+h)} = \sqrt{6-2x-2h}$$

$$\frac{g(x+h)-g(x)}{h} = \frac{\sqrt{6-2x-2h} - \sqrt{6-2x}}{h} \cdot \frac{\sqrt{6-2x-2h}+\sqrt{6-2x}}{\sqrt{6-2x-2h}+\sqrt{6-2x}}$$

$$= \frac{(6-2x-2h)-(6-2x)}{h\left(\sqrt{6-2x-2h}+\sqrt{6-2x}\right)} = \frac{-2\cancel{h}}{\cancel{h}\left(\sqrt{6-2x-2h}+\sqrt{6-2x}\right)} = \boxed{\frac{-2}{\sqrt{6-2x-2h}+\sqrt{6-2x}}}$$

11. For the following function $g(x) = \dfrac{8x^2-10x+3}{x-1}$ find Vertical, Horizontal and Slant asymptote(s).

$$g(x) = \frac{(2x-1)(4x-3)}{x-1}$$

$\Rightarrow$ Vertical Asymptote at $x=1$

Horizontal Asymptote ? $\dfrac{8x^2-10x+3}{x-1} = \dfrac{\frac{8x^2}{x} - \frac{10x}{x} + \frac{3}{x}}{\frac{x}{x} - \frac{1}{x}} = \dfrac{8x - 10 + \frac{3}{x}}{1 - \frac{1}{x}}$

as $x \to \infty$ $\quad \frac{1}{x} \to 0$, $\frac{3}{x} \to 0$ and $8x \to \infty$ therefore $g(x) \to \infty$ No Horizontal Asy.

Slant Asymptote:

$$\begin{array}{r|l} & 8x-2 \\ \hline x-1 & 8x^2-10x+3 \\ & -(8x^2-8x) \\ \hline & -2x+3 \\ & -(-2x+2) \\ \hline & 1 \end{array}$$

$\Rightarrow \dfrac{8x^2-10x+3}{x-1} = 8x-2+\dfrac{1}{x-1}$

slant asymptote $y = 8x-2$

12. $f(x) = \dfrac{2x^2 - 7x + 3}{x^2 - 2x - 3} = \dfrac{(2x-1)\cancel{(x-3)}}{\cancel{(x-3)}(x+1)}$

Domain: $(-\infty,-1) \cup (-1,3) \cup (3,\infty)$

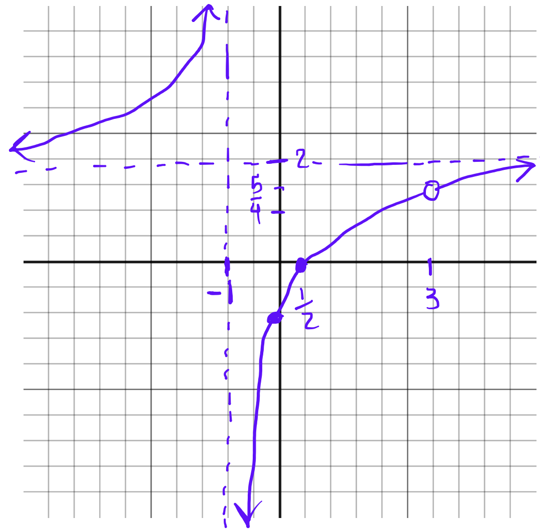
Hole(s): $(3, 5/4)$

Vertical Asymptote(s): $x = -1$

y-intercept: $(0,-1)$

x-intercept(s): $(1/2, 0)$, ~~$(3,0)$~~ not in dom.

Horizontal Asymptote(s): $y = 2$



Work:

Hole at $x = 3 \longrightarrow \dfrac{6-1}{3+1} = \dfrac{5}{4}$ $\quad (3, 5/4)$

$f(x)$ sign chart: $+$ at -1, $-$ at 0, $-$, $+$ at $\frac{1}{2}$

y-int. let $x=0 \longrightarrow \dfrac{2(0)-1}{0+1} = \dfrac{-1}{1} = -1$

x-int let $y=0 \longrightarrow (2x-1)(x-3)=0 \longrightarrow x=3$ & $x=\frac{1}{2}$

Vertical asy. $x=-1 \rightarrow$ odd multiplicity $\rightarrow$ or

Check if $x \longrightarrow -1^-$ (from left) e.g. $-1.001 \Rightarrow \dfrac{2(-1.001)-1}{-1.001+1} = \dfrac{-}{-} > 0 \Rightarrow y \to +\infty$

if $x \to -1^+$ (from right) e.g. $-0.999 \Rightarrow \dfrac{2(-0.999)-1}{-0.999+1} = \dfrac{-}{+} < 0 \Rightarrow y \to -\infty$

Sign chart: $+$ at -1.001, -1, $-$ at -0.999

Horizontal Asymptote:
$$\frac{\frac{2x^2}{x^2} - \frac{7x}{x^2} + \frac{3}{x^2}}{\frac{x^2}{x^2} - \frac{2x}{x^2} - \frac{3}{x^2}} = \frac{2 - \frac{7}{x} + \frac{3}{x^2}}{1 - \frac{2}{x} - \frac{3}{x^2}}$$

as $x \to \infty$ $\frac{7}{x}$, $\frac{3}{x^2}$, $\frac{2}{x}$ & $\frac{3}{x^2}$ all approach zero so $f(x) \to \dfrac{2-0}{1-0} = 2$

$\longrightarrow \boxed{y=2}$

13. Solve for v in the following equation.

$$\left|\frac{(v+4)(v+5)}{v^2-1}\right| = 1$$

Domain:
$v \in (-\infty,-1)\cup(-1,1)\cup(1,\infty)$

1) $$\frac{(v+4)(v+5)}{v^2-1} = 1$$

$$v^2 + 9v + 20 = v^2 - 1$$

$$9v + 20 = -1$$

$$9v = -21$$

$$\boxed{v = -\frac{7}{3}}$$

2) $$\frac{(v+4)(v+5)}{v^2-1} = -1$$

$$v^2 + 9v + 20 = -v^2 + 1$$

$$2v^2 + 9v + 19 = 0$$

$$v = \frac{-9 \pm \sqrt{81 - 4(2)(19)}}{4}$$

$$v = \frac{-9 \pm \sqrt{81 - 152}}{4}$$

No solution

14. For the following function, state the domain, identify the intercepts, analyze the end behavior and sketch the graph.

$$h(x) = \sqrt{x+3}\,(1+3x)^{\frac{-1}{5}} = \frac{\sqrt{x+3}}{(1+3x)^{\frac{1}{5}}}$$

$x \geq -3$ $\quad$ $1+3x \neq 0 \rightarrow x \neq -\frac{1}{3}$

$-3 \quad -\frac{1}{3}$

Domain: $x \in [-3, -\frac{1}{3}) \cup (-\frac{1}{3}, \infty)$

X-int: $x=0 \rightarrow h(0) = \frac{\sqrt{3}}{1} = \sqrt{3}$ $\quad (0, \sqrt{3})$

Y-int.: $y=0 \rightarrow 0 = \frac{\sqrt{x+3}}{(1+3x)^{1/5}} \rightarrow \sqrt{x+3} = 0 \rightarrow x = -3$ $\quad (-3, 0)$

Vertical Asy. at $x = -\frac{1}{3}$

$+ \quad - \quad +$

$\frac{-1}{2} \quad -\frac{1}{3} \quad \frac{-1}{5}$

$\frac{\sqrt{-\frac{1}{2}+3}}{(1-\frac{3}{2})^{\frac{1}{5}}} = \frac{+}{-} < 0$ so as $x \rightarrow -\frac{1}{3}^+$ $f(x) \rightarrow +\infty$

$\frac{\sqrt{-\frac{1}{5}+3}}{(1-\frac{3}{5})^{\frac{1}{5}}} = \frac{+}{+} > 0$ as $x \rightarrow -\frac{1}{3}^-$, $f(x) \rightarrow -\infty$

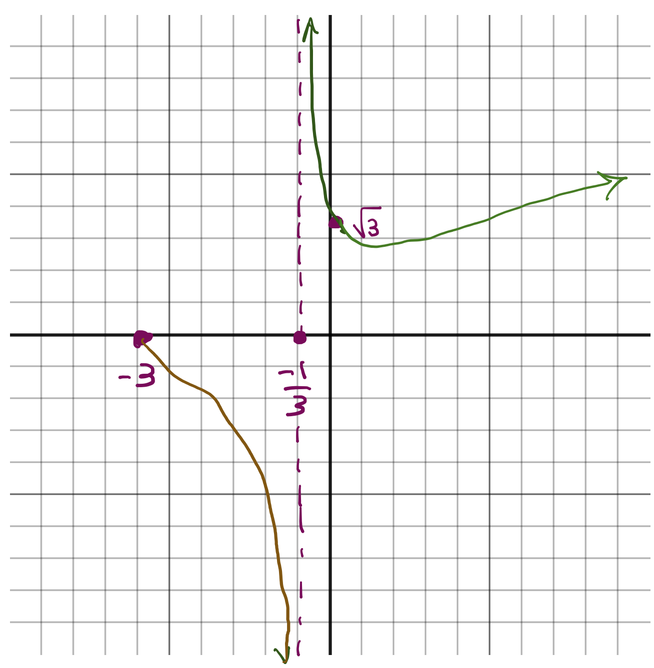
End behavior

$$\frac{\sqrt{x+3}}{(1+3x)^{\frac{1}{5}}} = \frac{\frac{\sqrt{x+3}}{(3x)^{\frac{1}{5}}}}{\left(\frac{1}{3x}+1\right)^{\frac{1}{5}}} = \frac{\sqrt{\frac{x}{(3x)^{2/5}} + \frac{3}{(3x)^{\frac{2}{5}}}}}{\left(\frac{1}{3x}+1\right)^{\frac{1}{5}}}$$

$$= \frac{\sqrt{\frac{x^{3/5}}{(3)^{2/5}} + \frac{3}{(3x)^{\frac{2}{5}}}}}{\left(\frac{1}{3x}+1\right)^{\frac{1}{5}}}$$

as $x \rightarrow \infty$ $\quad \frac{3}{(3x)^{\frac{2}{5}}}$ & $\frac{1}{3x}$ approach zero

but $\frac{x^{3/5}}{3^{2/5}} \longrightarrow \infty$



15. Solve $\sqrt{t^4+9} = \sqrt{6}\,t$ for t.

square both sides

$\to t^4 + 9 = 6t^2$

$t^4 - 6t^2 + 9 = 0$

$(t^2 - 3)^2 = 0$

$t^2 = 3 \longrightarrow t = \sqrt{3}$

$t = -\sqrt{3}$

16. Find the intervals where the following inequality is true. $2x\,(2x-3)^{-2} \le 4\,(2x-3)^{-3}$

Domain restriction $2x - 3 \neq 0$ $\quad x \neq \frac{3}{2}$

$x \in (-\infty, \frac{3}{2}) \cup (\frac{3}{2}, \infty)$

$$\frac{2x}{(2x-3)^2} \le \frac{4}{(2x-3)^3}$$

$$\frac{2x}{(2x-3)^2} - \frac{4}{(2x-3)^3} \le 0$$

$$y = \frac{2x}{(2x-3)^2} - \frac{4}{(2x-3)^3} = 0$$

multiply by LCD $\Rightarrow$ $2x\,(2x-3) - 4 = 0$

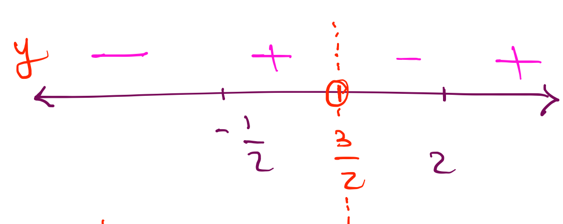
$4x^2 - 6x - 4 = 0$

$2\,(\underbrace{2x^2 - 3x - 2}) = 0$

$\frac{1}{2} \quad -3 \quad -4 \quad \frac{-4}{2} = -2$

$(2x+1)(x-2) = 0$

$x = -\frac{1}{2}\,, \quad x = 2$



we want $y \le 0$

$x \in (-\infty, -\frac{1}{2}] \cup (\frac{3}{2}, 2]$

17. Given $f(x) = \dfrac{-3x+4}{x-2}$ is a one-to-one function, compute $f^{-1}(x)$ and state domain and range of $f(x)$ and $f^{-1}(x)$.

Domain of $f(x)$ $\quad x \in (-\infty, +2) \cup (2, \infty) \Rightarrow$ Range of $f^{-1}(x)$

Range of $f(x)$ $\quad x \in (-\infty, -3) \cup (-3, \infty) \Leftarrow$ Domain of $f^{-1}(x)$

$$y = \frac{-3x+4}{x-2} \rightarrow yx - 2y = -3x+4 \rightarrow yx + 3x = 4 + 2y$$

$$x(y+3) = 4 + 2y$$

$$x = \frac{4+2y}{y+3}$$

$$\Rightarrow f^{-1}(x) = \frac{4+2x}{x+3}$$

Check both $f \circ f^{-1}(x) \stackrel{?}{=} x$ & $f^{-1} \circ f(x) \stackrel{?}{=} x$

$$(f \circ g)(x) = \frac{-3\left(\frac{2x+4}{x+3}\right) + 4}{\frac{2x+4}{x+3} - 2} = \frac{\frac{-6x-12}{x+3} + \frac{4x+12}{x+3}}{\frac{2x+4-2x-6}{x+3}} = \frac{\frac{-2x}{x+3}}{\frac{-2}{x+3}} =$$

$$\frac{-2x}{\cancel{x+3}} \cdot \frac{\cancel{x+3}}{-2} = \frac{-2x}{-\cancel{2}} = \boxed{x} \checkmark$$

Checking the other side:

$$(g \circ f)(x) = \frac{2\left(\frac{-3x+4}{x-2}\right) + 4}{\left(\frac{-3x+4}{x-2}\right) + 3} = \frac{\frac{-6x+8}{x-2} + 4}{\frac{-\cancel{3x}+4+\cancel{3x}-6}{x-2}} = \frac{\frac{-6x+\cancel{8}+4x-\cancel{8}}{x-2}}{\frac{-2}{x-2}} = \frac{-2x}{\cancel{x-2}} \cdot \frac{\cancel{x-2}}{-2}$$

$$= \boxed{x} \checkmark$$

18. Describe the transformation(s) of the graph of $f(x) = 3^x$ that yield(s) the graph of $g(x) = 3^{-0.7x} + 1$, then choose the graph that matches the function.

Transformations:

① Horizontal stretch (by $\frac{1}{0.7}$ units)

② Reflect over y-axis

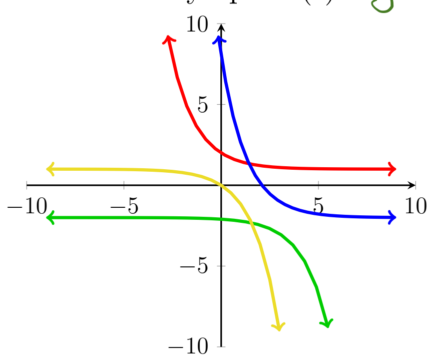
③ Vertical shift up 1

Domain: $x \in (-\infty, \infty)$

x-intercept(s): None $3^{-0.7x} = -1 \rightarrow \ln(3)^{-0.7x} = \ln(-1) \rightarrow$ No solution

y-intercept(s): let $x=0$ $g(0) = 1+1 = 2$ $(0,2)$

Horizontal Asymptote(s): $y=1$



red graph ✓

If $g(x)$ is composition of two functions, $f(x) = 3^x$ and $h(x)$ such that $g(x) = h(f(-0.7x))$. Find $h(x)$.

$f(x) = 3^x \rightarrow f(-0.7x) = 3^{-0.7x}$

$h(x) = x+1$

$h(f(-0.7x)) = h(3^{-0.7x}) = 3^{-0.7x} + 1$

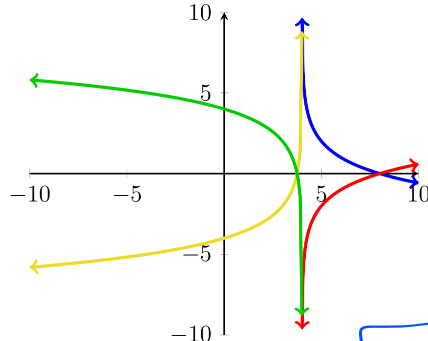
19. Describe the transformations of $f(x) = \log_2(x)$ that yield $g(x) = -\log_2(x-4)+2$. Then state the domain, x-intercept, and vertical asymptote of the logarithmic function $f(x)$, then choose the graph that matches the function.

Transformations:

① Horizontal shift right 4

② Reflection w.r.t x-axis

③ vertical shift up 2 units



Blue graph

Domain: $x-4>0$ $\quad x>4$

$x \in (4, \infty)$

x-intercept(s):

let $y=0$ $\quad 2 = \log_2(x-4) \iff 2^2 = x-4 \rightarrow x = 8$ $\quad (8,0)$

y-intercept(s): let $x=0$ (we can't since zero is not in the domain)

Vertical Asymptote(s):

$X=4$ (found by checking the transformations.)

20. Solve $\dfrac{15}{100+e^{2x}} = 3$ for x. **Always check for extraneous solutions.**

(Domain restriction $e^{2x} + 100 = 0$
$e^{2x} = -100 \longrightarrow$ No solution. since $e^{2x} > 0$)

$15 = 3(100 + e^{2x})$

$5 = 100 + e^{2x}$

$-95 = e^{2x} \longrightarrow$ No solutions

[note: $\ln(e^{2x}) = \ln(\overbrace{-95}^{\text{not in domain}})$]

21. Use the properties of logarithms to expand the expression as a sum, difference, and/or constant multiple of logarithms. (Assume all variables are positive.)

$$\ln \sqrt[3]{\frac{x^2}{x^2 - 8x - 20}} = \ln\left(\frac{x^2}{(x-10)(x+2)}\right)^{\frac{1}{3}}$$

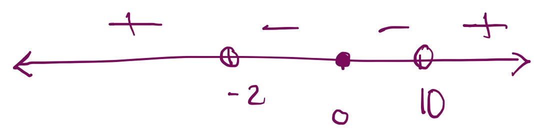
domain?

$$\frac{x^2}{(x-10)(x+2)} > 0$$

&

$x \neq 10$

$x \neq -2$



$x \in (-\infty, -2) \cup (10, \infty)$

$$\ln \sqrt[3]{\frac{x^2}{x^2-8x-20}} = \ln\left(\frac{x^2}{x^2-8x-20}\right)^{\frac{1}{3}} = \frac{1}{3}\left[\ln\left(\frac{x^2}{x^2-8x-20}\right)\right]$$

$$= \frac{1}{3}\left[\ln x^2 - \ln(x^2-8x-20)\right] = \frac{1}{3}\left[2\ln x - \ln\left((x-10)(x+2)\right)\right]$$

$$= \frac{1}{3}\left[2\ln x - \left(\ln(x-10) + \ln(x+2)\right)\right] = \frac{1}{3}\left(2\ln x - \ln(x-10) - \ln(x+2)\right)$$