Math 308: Week-in-Review 1

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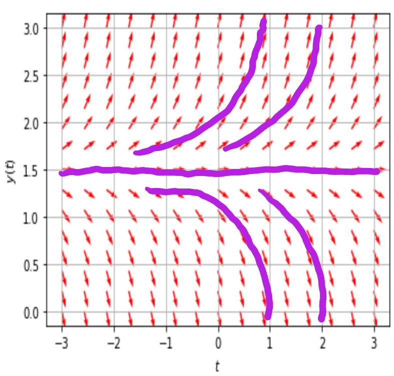
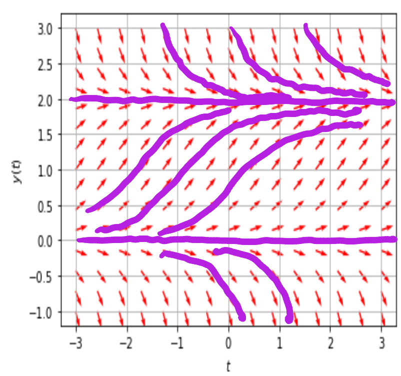
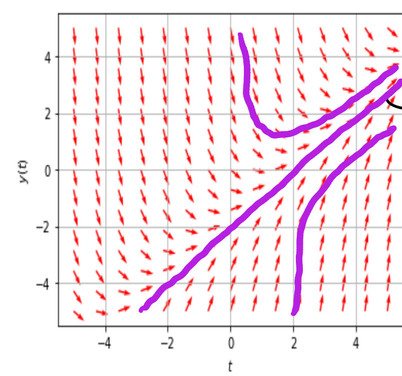
1. Given the following differential equations and their corresponding direction field, determine the behavior as t increases.



(a) $y' = 2y - 3$ equilibrium



(b) $y' = y(2-y)$



linear solution

Solutions asymptote towards the linear solution

(c) $y' = t - 1 - y$

(a)

$y(0) = 3/2,\ y(t) = 3/2$

$y(0) > 3/2,\ y(t) \to \infty$

$y(0) < 3/2,\ y(t) \to -\infty$

(b)

$y(0) = 0 \Rightarrow y(t) = 0$ equil. solution

$y(0) = 2 \Rightarrow y(t) = 2$

$y(0) > 2 \Rightarrow y(t) \to 2$

$0 < y(0) < 2 \Rightarrow y(t) \to 2$

$y(0) < 0 \Rightarrow y(t) \to -\infty$

(c)

$y(t) \to \infty$ as $t \to \infty$

$y(t) \sim at + b$ for some linear function as $t \to \infty$

Find the equation of the linear solution of the last differential equation.

$y = at + b,\quad y' = a$

$y' = t - 1 - y$

$a = t - 1 - (at + b)$

$= (-b - 1) + (1 - a)t$

$a = -b - 1,\quad 0 = 1 - a \Rightarrow a = 1$

$-b - 1 = 1 \Rightarrow b = -2$

$$\boxed{y = t - 2}$$

2. Given the differential equation $\dfrac{dy}{dt} = ty - 1$

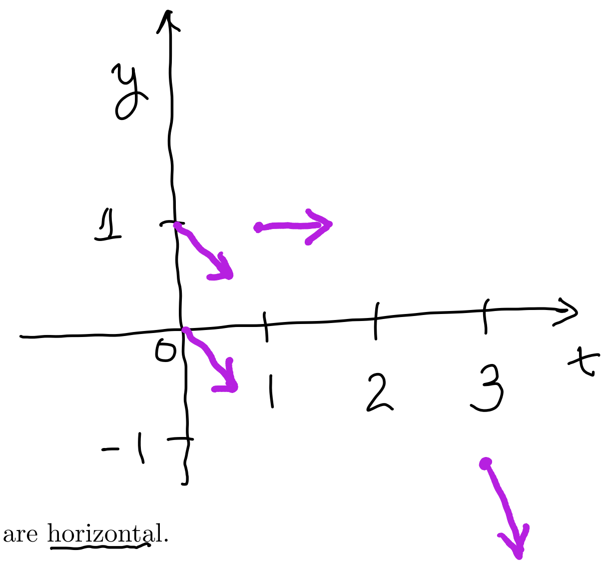
   (a) What is the slope of the graph of the solutions at the point $(0,1)$, at the point $(1,1)$ at the point $(3,-1)$ and at the point $(0,0)$?

slope @ $(0,1)$: $(0)(1) - 1 = -1$

slope @ $(1,1)$: $(1)(1) - 1 = 0$

slope @ $(3,-1)$: $(3)(-1) - 1 = -4$

slope @ $(0,0)$: $(0)(0) - 1 = -1$



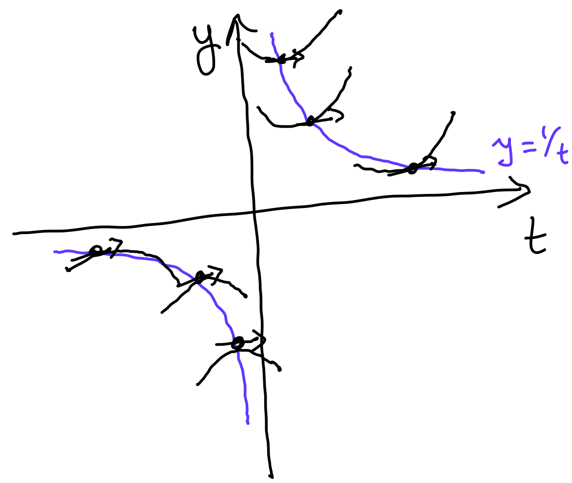
   (b) Find all points where the tangents to the solution curves are horizontal.

$y' = 0$, $y' = ty - 1 = 0$

$ty = 1$

$y = 1/t$

$$\boxed{\{(t, 1/t) : t \in \mathbb{R} \setminus \{0\}\}}$$



min, max

   (c) Describe the nature of the critical points of the solution curves.

$y' = 0$

$y' = ty - 1 \Rightarrow y'' = y + ty'$ → product rule

→ critical point

$= y + t(0)$

$= y$

* $y'' = y$

* $y'' > 0$ if $y > 0$ (concave up, minimum)

$y'' < 0$ if $y < 0$ (concave down, maximum)

3. The instantaneous rate of change of the temperature T of coffee at time t is proportional to the difference between the temperature M of the air and the temperature T of coffee at time t.

(a) Find the mathematical model for the problem.

Newton's Law of Cooling.

M room temperature

$$\frac{dT}{dt} = -k(T-M) \qquad k>0: \text{ physical constant}$$

(b) Given that the room temperature is 75°F (M) and $k = 0.08$, find the solutions to the differential equation.

$$\frac{\frac{dT}{dt}}{T-M} = -k$$

$$\frac{d}{dt}\ln|T-M| = -k$$

integrate $\ln|T-M| = -kt + c$

exponentiate $|T-M| = e^{-kt+c} = e^{c}\cdot e^{-kt}$

$$T-M = \pm e^{C}\cdot e^{-kt} = De^{-kt} \qquad \text{where } D = \pm e^{C} \text{ arbitrary}$$

$$T = M + De^{-kt} \qquad M = 75°F,\ k = 0.08$$

$$\boxed{T = 75 + De^{-0.08t}}$$

(c) The initial temperature of the coffee is 200°F. Find the solution to the problem.

$T(0) = 200°F$. Find D.

$$200 = 75 + De^{0} = 75 + D \Rightarrow D = 200 - 75 = 125$$

$$\boxed{T = 75 + 125e^{-0.08t}}$$

4. Your swimming pool containing 60,000 gallons of water has been contaminated by 5 kg of a non-toxic dye that leaves a swimmer's skin an unattractive green. The pool's filtering system can take water from the pool, remove the dye, and return the water to the pool at a flow rate of 200 gallons per minute.

   (a) Write down the initial value problem for the filtering process, let $q(t)$ be the amount of dye in the pool at any time t.

ODE + IC

$$\frac{dq}{dt} = -(\text{concentration}) * (\text{flow rate}) = -\frac{q(t)}{60{,}000} * 200$$

↳ flow rate

↳ concentration

$$\begin{cases} \dfrac{dq}{dt} = -\dfrac{1}{300} q(t) \\ q(0) = 5\text{ kg} \end{cases}$$

   (b) Solve the problem.

$$\frac{dq}{dt} = -\frac{1}{300} q(t)$$

$$\frac{1}{q(t)} \frac{dq}{dt} = -\frac{1}{300}$$

$$\frac{d}{dt} \ln|q(t)| = -\frac{1}{300}$$

integrate: $\ln|q(t)| = \int -\frac{1}{300}\,dt = -\frac{1}{300}t + k$

$$|q(t)| = e^{-\frac{1}{300}t + k} = e^k \cdot e^{-\frac{1}{300}t}$$

$$q(t) = \pm e^k \cdot e^{-\frac{1}{300}t}, \quad C = \pm e^k$$

$$q(t) = Ce^{-\frac{1}{300}t}$$

$q(0) = 5 \Rightarrow C = 5$

$$\boxed{q(t) = 5e^{-\frac{1}{300}t}}$$

   (c) You have invited several dozen friends to a pool party that is scheduled to begin in 4 hours. You have also determined that the effect of the dye is imperceptible if its concentration is less than 0.02 grams per gallon. Is your filtering system capable of reducing the dye concentration to this level within 4 hours?

4 hrs → $4 * 60 = 240$ mins

$$q(240) = 5e^{-\frac{1}{300}\cdot 240} = 5e^{-4/5} = 2.2466\text{ kg}$$

dye after 4 hrs

concentration: at 4 hrs

$$\frac{q(240)}{60{,}000} = \frac{2.2466\text{ kg}}{60{,}000\text{ gal}} = 3.744 \times 10^{-5}\text{ kg/gal}$$

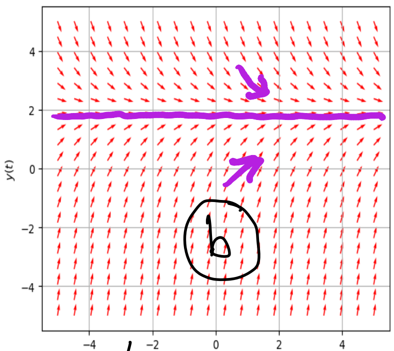
$$= 3.744 \times 10^{-2}\text{ g/gal}$$

$$= 0.03744\text{ g/gal} > 0.02\text{ g/gal}$$

NOT ENOUGH TIME

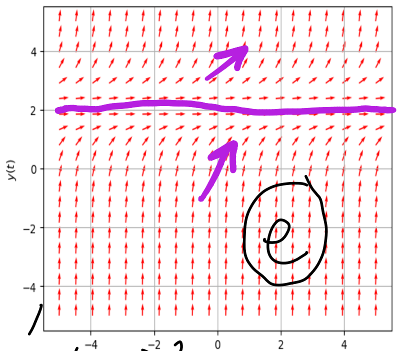
5. Match the direction field to the differential equations

$(a)\ y' = y - 2 \quad (b)\ y' = 2 - y \quad (c)\ y' = 2 + y \quad (d)\ y' = -2 - y \quad (e)\ y' = (y-2)^2 \quad (f)\ y' = (y+2)^2$



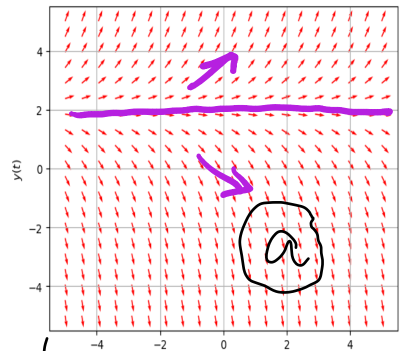
$y' = 2 - y$

(a) Direction field 1



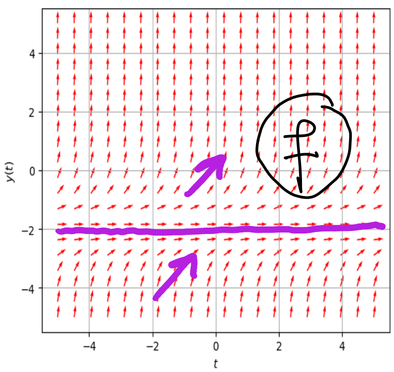
$y' = (y-2)^2$

(b) Direction field 2



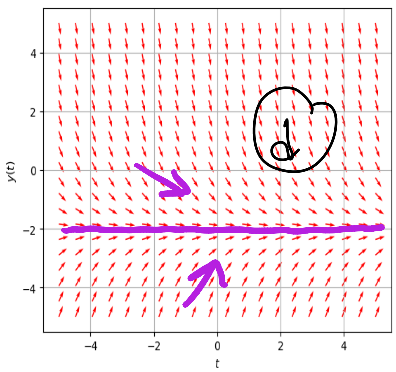
$y' = y - 2$

(c) Direction field 3



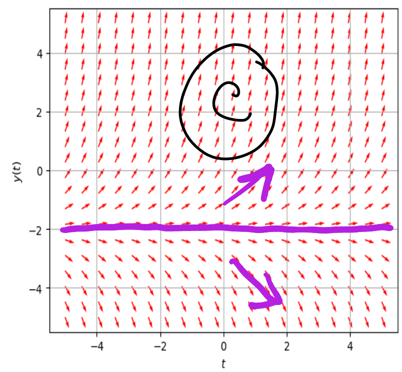
(a) Direction field 4

$y' = (y+2)^2$



(b) Direction field 5

$y' = -2 - y$



(c) Direction field 6

$y' = 2 + y$

6. Given the following differential equations, classify each as an ordinary differential equation, partial differential equation, give the order. If the equation is an ordinary differential equation, say whether the equation is linear or nonlinear.

   (a) $\dfrac{dy}{dx} = 3y + x^2$ ODE, 1st order, linear

   (b) $5\dfrac{d^4y}{dx^4} + y = x(x-1)$ ODE, 4th order, linear

   (c) $\dfrac{\partial N}{\partial t} = \dfrac{\partial^2 N}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial N}{\partial r} + kN$ PDE, 2nd order, linear

   (d) $\dfrac{dx}{dt} = x^2 - t$ ODE, 1st order, nonlinear

   (e) $2t\dfrac{d^3y}{dt^3} - 5y\sin(t) = 5t^2y$ ODE, 3rd order, linear

7. (a) Show that $f(x) = (x^2 + Ax + B)e^{-x}$ is a solution to

$$y'' + 2y' + y = 2e^{-x}$$

   for all real numbers A and B.

$$y = (x^2 + Ax + B)e^{-x}, \quad y' = (2x + A)e^{-x} - (x^2 + Ax + B)e^{-x}$$

$$y'' = 2e^{-x} - (2x + A)e^{-x} - (2x + A)e^{-x} + (x^2 + Ax + B)e^{-x}$$

$$\begin{aligned} y'' + 2y' + y &= 2e^{-x} - 2\cancel{(2x + A)e^{-x}} + (x^2 + \cancel{Ax} + B)e^{-x} \\ &\quad + 2\cancel{(2x + A)e^{-x}} - 2(\cancel{x^2 + Ax} + B)e^{-x} + (x^2 + \cancel{Ax} + B)e^{-x} \\ &= 2e^{-x} \checkmark \end{aligned}$$

   (b) Find a solution that satisfies the initial condition $y(0) = 3$ and $y'(0) = 1$.

$$y(x) = (x^2 + Ax + B)e^{-x}, \quad y(0) = B = 3$$

$$y'(x) = (2x + A)e^{-x} - (x^2 + Ax + B)e^{-x}$$

$$y'(0) = A - B = 1 \Rightarrow A = B + 1 = 4$$

$$\boxed{y(x) = (x^2 + 4x + 3)e^{-x}}$$

8. Determine for which values of r the function t^r is a solution of the differential equation

Euler eqn

$$t^2y'' - 4ty' + 4y = 0, \quad t > 0$$

$y = t^r$, $y' = rt^{r-1}$, $y'' = r(r-1)t^{r-2}$

$\underbrace{t^2\left[r(r-1)t^{r-2}\right] - 4trt^{r-1} + 4t^r}_{t^2y'' - 4ty' + 4y} = r(r-1)t^r - 4rt^r + 4t^r$

$= t^r\left[r^2 - r - 4r + 4\right]$

$= t^r\left[r^2 - 5r + 4\right]$

$= 0$ for all t

$r^2 - 5t + 4 = 0$

$(r-4)(r-1) = 0 \Rightarrow r = 1, 4$

$$\boxed{\begin{cases} y_1(t) = t \\ y_2(t) = t^4 \end{cases}}$$

9. For which values of r is the function $(x-1)e^{-rx}$ solution to $y'' - 6y' + 9y = 0$?

$y = (x-1)e^{-rx}$, $y' = e^{-rx} - r(x-1)e^{-rx} = e^{-rx}\left[1 - rx + r\right]$

$y'' = \left[-r - r(1 - rx + r)\right]e^{-rx} = \left[r^2x - 2r - r^2\right]e^{-rx}$

$y'' - 6y' + 9y = \left[r^2x - 2r - r^2 - 6 + 6rx - 6r + 9x - 9\right]e^{-rx}$

$= (r^2 + 6r + 9)x - (r^2 + 8r + 15) = 0$ for all x

$r^2 + 6r + 9 = 0$

$(r+3)^2 = 0$, $r = -3$ ✓

$r^2 + 8r + 15 = 0$

$(r+5)(r+3) = 0$

$r = -3$ ✓

$$\boxed{y = (x-1)e^{3x}}$$

10. Consider the second order differential equation

$$y'' + 4y = 0$$

(a) Show that the function $y = c_1 \sin(2x) + c_2 \cos(2x)$ is a solution of the differential equation.

$y = c_1 \sin(2x) + c_2 \cos(2x)$, $y' = 2c_1 \cos(2x) - 2c_2 \sin(2x)$

$y'' = -4c_1 \sin(2x) - 4c_2 \cos(2x)$

$= -4\left[c_1 \sin(2x) + c_2 \cos(2x)\right]$

$= -4y$

$$\boxed{y'' + 4y = 0}$$

(b) Suppose the initial conditions are provided as $y(0) = 0$ and $y'(0) = 2$. Determine the values of c_1 and c_2.

$y(x) = c_1 \sin(2x) + c_2 \cos(2x)$

$y(0) = c_2 = 0$, $y'(x) = 2c_1 \cos(2x) - 2c_2 \sin(2x)$

$y'(0) = 2c_1 = 2 \Rightarrow c_1 = 1$

$$\boxed{y(x) = \sin(2x)}$$

