Math 308: Week-in-Review 7

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* definition *

1. Use the definition to find the Laplace transforms of $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

   (a) $f(t) = e^{at}$, where a is a nonzero real number.

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at} = \int_0^\infty e^{(a-s)t}\,dt = \lim_{A\to\infty} \int_0^A e^{(a-s)t}\,dt$$

$$= \lim_{A\to\infty} \left\{ \frac{1}{(a-s)} e^{(a-s)t} \Big|_{t=0}^{t=A} \right\} = \lim_{A\to\infty} \left\{ \frac{1}{a-s} \left[e^{(a-s)A} - 1 \right] \right\}$$

$$= \begin{cases} \frac{1}{s-a}, & \text{if } s > a \quad (a-s<0) \\ \infty, & \text{if } s \leq a \end{cases}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \quad (\text{provided } s > a)$$

   (b) $f(t) = \cos(bt)$, where b is a nonzero real number.

$$\mathcal{L}\{\cos(bt)\} = \int_0^\infty e^{-st} \cos(bt)\,dt = \lim_{A\to\infty} \int_0^A e^{-st} \cos(bt)\,dt$$

$$= \lim_{A\to\infty} \left\{ \frac{e^{-st}}{s^2+b^2} \left(-s\cos(bt) + b\sin(bt) \right) \Big|_{t=0}^{t=A} \right\}$$

$$= \lim_{A\to\infty} \left\{ \frac{e^{-sA}}{s^2+b^2} \left(-s\cos(bA) + b\sin(bA) \right) - \frac{1}{s^2+b^2}(-s) \right\}$$

$$= \begin{cases} \frac{s}{s^2+b^2}, & \text{if } s > 0 \\ \infty, & \text{if } s < 0 \end{cases}$$

$$\mathcal{L}\{\cos(bt)\} = \frac{s}{s^2+b^2}, \text{ if } s > 0$$

(c) $f(t) = \begin{cases} 2t+1, & 0 \le t < 2, \\ 3t, & t \ge 2. \end{cases}$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^2 e^{-st}(2t+1)\,dt + \int_2^\infty e^{-st}\cdot 3t\,dt$$

$$= 2\int_0^2 t e^{-st}\,dt + \int_0^2 e^{-st}\,dt + 3\int_2^\infty t e^{-st}\,dt$$

integration by parts

	u	dv
	t	e^{-st}
$+$	1	$-\frac{1}{s}e^{-st}$
$-$	0	$\frac{1}{s^2}e^{-st}$

$$= 2\left\{ -\frac{te^{-st}}{s} - \frac{1}{s^2}e^{-st} \right\}_{t=0}^{t=2} + \left\{ \frac{1}{-s}e^{-st} \right\}_{t=0}^{t=2}$$

$$(*) \; + 3\lim_{A\to\infty}\left\{ -\frac{te^{-st}}{s} - \frac{1}{s^2}e^{-st} \right\}_{t=2}^{t=A}$$

$$= -\frac{4e^{-2s}}{s} - \frac{2}{s^2}e^{-2s} + \frac{2}{s^2} - \frac{e^{-2s}}{s} + \frac{1}{s} + \frac{6}{s}e^{-2s} + \frac{3e^{-2s}}{s^2}$$

$$\mathcal{L}\{f(t)\} = \frac{e^{-2s}}{s} + \frac{e^{-2s}}{s^2} + \frac{2}{s^2} + \frac{1}{s}$$

$(*)$ since $-\frac{A}{s}e^{-sA} - \frac{1}{s^2}e^{-sA} \to 0$ as $A \to \infty$ if $s > 0$

(d) $f(t) = t$

$$\mathcal{L}\{t\} = \int_0^\infty t\, e^{-st}\, dt$$

$$= \lim_{A\to\infty} \int_0^A t\, e^{-st}\, dt$$

$$= \lim_{A\to\infty} \left[-\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st} \right]_{t=0}^{t=A}$$

$$= \lim_{A\to\infty} \left\{ \left(-\frac{A}{s} e^{-sA} - \frac{1}{s^2} e^{-sA} \right) - \left(-\frac{1}{s^2} \right) \right\}$$

	t	e^{-st}
$+$	1	$-\frac{1}{s} e^{-st}$
$-$	0	$\frac{1}{s^2} e^{-st}$

$\mathcal{L}\{t\} = \frac{1}{s^2}$ if $s > 0$

(e) $f(t) = t^2$ $\quad \mathcal{L}\{t^2\} = \int_0^\infty t^2 e^{-st}\, dt = \lim_{A\to\infty} \int_0^A t^2 e^{-st}\, dt$

$$= \lim_{A\to 0} \left\{ -\frac{t^2}{s} e^{-st} - \frac{2t}{s^2} e^{-st} - \frac{2}{s^3} e^{-st} \right\}_{t=0}^{t=A}$$

$$= \lim_{A\to 0} \left\{ -\frac{A^2}{2} e^{-sA} - \frac{2A}{s^2} e^{-sA} - \frac{2}{s^3} e^{-sA} + \frac{2}{s^3} \right\}$$

	t^2	e^{-st}
$+$	$2t$	$-\frac{1}{s} e^{-st}$
$-$	2	$\frac{1}{s^2} e^{-st}$
$+$	0	$-\frac{1}{s^3} e^{-st}$

$\mathcal{L}\{t^2\} = \frac{2}{s^3}$ if $s > 0$

In general, $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, $s > 0$

$n = 0, 1, 2, 3, \dots$

2. Find the inverse Laplace transform of the following functions

(a) $F(s) = \dfrac{4}{(s-2)^5}$

$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ for $F(s) = \mathcal{L}\{f(t)\}$

$$\mathcal{L}^{-1}\left\{\frac{4!}{s^5}\right\} = t^4 \Rightarrow \mathcal{L}^{-1}\left\{\frac{4}{s^5}\right\} = \frac{1}{6}t^4$$

by the translation property, $\mathcal{L}^{-1}\left\{\dfrac{4}{(s-2)^5}\right\} = \dfrac{1}{6}t^4e^{2t}$

(b) $F(s) = \dfrac{8s^2-4s+12}{s(s^2+4)} = \dfrac{4(2s^2-s+3)}{s(s^2+4)} = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+4}$

$$4(2s^2-s+3) = A(s^2+4) + (Bs+C)s \Rightarrow A=3,\ B=5,\ C=-4$$

$$F(s) = \frac{3}{s} + \frac{5s-4}{s^2+4} = \frac{3}{s} + \frac{5s}{s^2+4} - \frac{4}{s^2+4}$$

$$= \frac{3}{s} + 5\frac{s}{s^2+4} - 2\frac{2}{s^2+4}$$

$$\mathcal{L}^{-1}\{F(s)\} = 3 + 5\cos(2t) - 2\sin(2t)$$

completing the square

(c) $F(s) = \dfrac{2s-3}{s^2+2s+10} = \dfrac{2s-3}{(s+1)^2+9} = \dfrac{2s-3}{(s+1)^2+3^2} = 2\dfrac{s+1}{(s+1)^2+3^2} - \dfrac{5}{(s+1)^2+3^2}$

$$F(s) = 2\frac{s+1}{(s+1)^2+3^2} - \frac{5}{3}\frac{3}{(s+1)^2+3^2}$$

$$\mathcal{L}^{-1}\{F(s)\} = 2e^{-t}\cos(3t) - \frac{5}{3}e^{-t}\sin(3t)$$

3. Use the Laplace transform to solve the initial value problem

$$y'' + 3y' + 2y = 4t, \quad y(0) = 1, \quad y'(0) = 0.$$

$$\mathcal{L}\{y'' + 3y' + 2y\} = \mathcal{L}\{4t\} \Rightarrow s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 3s\mathcal{L}\{y\} - 3y(0) + 2\mathcal{L}\{y\} = \mathcal{L}\{4t\}$$

$$\mathcal{L}\{y\}\left(s^2+3s+2\right) = \mathcal{L}\{4t\} + s + 3 \qquad \mathcal{L}\{4t\} = 4\mathcal{L}\{t\} = \frac{4}{s^2}$$

$$\mathcal{L}\{y\} = \frac{\mathcal{L}\{4t\}}{s^2+3s+2} + \frac{s+3}{s^2+3s+2}$$

$$= \frac{4}{s^2(s+2)(s+1)} + \frac{s+3}{(s+2)(s+1)}$$

* partial fractions *

$$\frac{4}{s^2(s+2)(s+1)} = \frac{A}{s^2} + \frac{B}{s+2} + \frac{C}{s+1} \Rightarrow 4 = A(s+2)(s+1) + Bs^2(s+1) + Cs^2(s+2)$$

$$\begin{cases} s=0: & 4 = 2A \Rightarrow A = 2 \\ s=-1: & 4 = C \Rightarrow C = 4 \\ s=-2: & 4 = -4B \Rightarrow B = -1 \end{cases}$$

$$\frac{s+3}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1} \Rightarrow s+3 = A(s+1) + B(s+2)$$

$$\begin{cases} s=-1: & 2 = B \Rightarrow B = 2 \\ s=-2: & 1 = -A \Rightarrow A = -1 \end{cases}$$

$$\mathcal{L}\{y\} = \frac{2}{s^2} - \frac{1}{s+2} + \frac{4}{s+1} - \frac{1}{s+2} + \frac{2}{s+1} = \frac{2}{s^2} - \frac{2}{s+2} + \frac{6}{s+1}$$

homogeneous solution

$$y(t) = \underbrace{2t}_{\text{particular solution}} \underbrace{- 2e^{-2t} + 6e^{-t}}_{\text{homogeneous solution}}$$

particular solution

4.

$$y'' + 9y = \cos 3t, \quad y(0) = 0, y'(0) = 1.$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 9\mathcal{L}\{y\} = \frac{s}{s^2+9}$$

$$(s^2+9)\mathcal{L}\{y\} = 1 + \frac{s}{s^2+9}$$

$$\mathcal{L}\{y\} = \frac{1}{s^2+9} + \frac{s}{(s^2+9)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} = \frac{1}{3}\sin(3t)$$

* recall $\mathcal{L}\{-t f(t)\} = \frac{d}{ds}\mathcal{L}\{f(t)\}$

$$\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = 3\frac{-2s}{(s^2+9)^2} = -6\frac{s}{(s^2+9)^2}$$

$$\Downarrow$$

$$\mathcal{L}\{-t\sin(3t)\} = -6\frac{s}{(s^2+9)^2}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{s}{(s^2+9)^2}\right\} = \frac{1}{6}t\sin(3t)$$

$$y(t) = \underbrace{\frac{1}{3}\sin(3t)}_{\text{homogeneous solution}} + \underbrace{\frac{1}{6}t\sin(3t)}_{\text{particular solution}}$$

(resonance)

5.

$$y'' - 2y' + 2y = e^{-t}, \quad y(0) = 0, \quad y'(0) = 1.$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) - 2\left(s\mathcal{L}\{y\} - y(0)\right) + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{-t}\}$$

$$(s^2 - 2s + 2)\mathcal{L}\{y\} = 1 + \frac{1}{s+1}$$

$s^2 - 2s + 2 = 0$ has complex roots $\Rightarrow$ complete the square

$$\mathcal{L}\{y\} = \frac{1}{s^2 - 2s + 2} + \frac{1}{(s+1)(s^2-2s+2)}$$

$$= \frac{1}{(s-1)^2+1} + \frac{1}{(s+1)(s^2-2s+2)}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\} = e^{t}\sin(t)$$

$\frac{1}{5}\mathcal{L}\{e^{-t}\}$

$$\frac{1}{(s+1)(s^2-2s+2)} = \frac{A}{s+1} + \frac{Bs+C}{s^2-2s+2} = \frac{1}{5(s+1)} + \frac{3-s}{4(s^2-2s+2)}$$

$$1 = A(s^2 - 2s + 2) + (Bs + C)(s+1)$$

$s = -1$: $1 = 5A \Rightarrow A = 1/5$

$s = 0$: $1 = 2A + C \Rightarrow 1 = \frac{2}{5} + C \Rightarrow C = 3/5$

$s = 1$: $1 = \frac{1}{5} + 2\left(B + \frac{3}{5}\right) \Rightarrow B = -1/5$

$$\frac{3-s}{5(s^2-2s+2)} = \frac{3-s}{5\left((s-1)^2+1\right)} = -\frac{1}{5}\,\frac{s-1}{(s-1)^2+1} + \frac{2}{5}\,\frac{1}{(s-1)^2+1}$$

$-\frac{1}{5}\mathcal{L}\{e^{t}\cos t\}$ $\quad$ $\frac{2}{5}\mathcal{L}\{e^{t}\sin t\}$

$$y(t) = e^{t}\sin(t) + \frac{1}{5}e^{-t} - \frac{1}{5}e^{t}\cos(t) + \frac{2}{5}e^{t}\sin(t)$$

$$y(t) = \frac{7}{5}e^{t}\sin(t) - \frac{1}{5}e^{t}\cos(t) + \frac{1}{5}e^{-t}$$