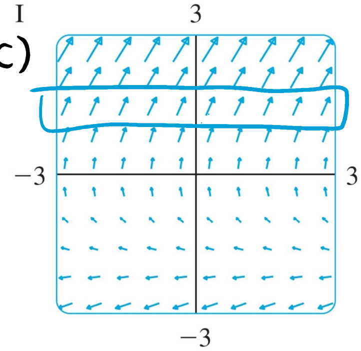
NOTE ♯8: SECTIONS 16.1-16.3

Problem 1. Match the vector fields F with the plots labeled I-IV. Give reasons for your choices.
(a) $\mathbf{F}(x, y) = \langle x, -y\rangle$ (b) $\mathbf{F}(x, y) = \langle y, \boxed{x - y}\rangle$ 0 (c) $\mathbf{F}(x, y) = \langle y, y + 2\rangle$
(d) $\mathbf{F}(x, y) = \langle \cos(x + y), x\rangle$

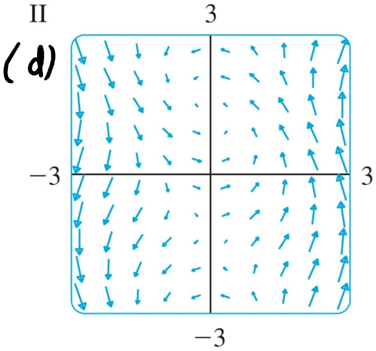
$x - y = 0 \Leftrightarrow y = x$

$(x, y) : \vec{F}(x,y) = \langle x, -y\rangle$

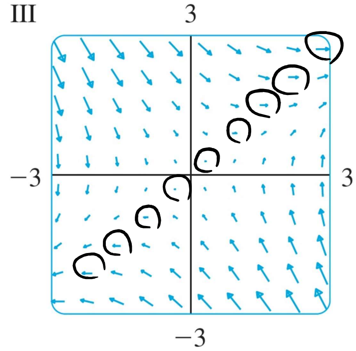
I (c)



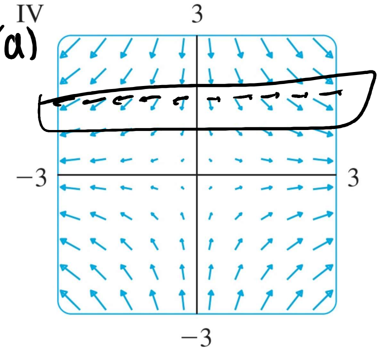
II (d)



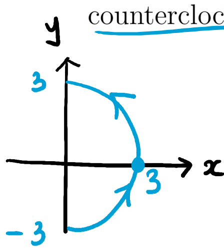
III (b)



IV (a)



Problem 2. Evaluate $\displaystyle\int_C xy^2 ds$, where C is the right half of the circle $x^2 + y^2 = 9$, oriented counterclockwise.



$C : \vec{r}(t) = \langle x(t), y(t)\rangle$

$\begin{cases} x(t) = 3\cos t \\ y(t) = 3\sin t \end{cases}$,

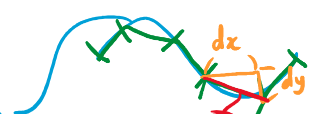
$-\frac{\pi}{2} \le t \le \frac{\pi}{2}$

$$\int_C xy^2 ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (3\cos t)(3\sin t)^2 \; 3\,dt$$

$$\left(\begin{aligned} ds &= \sqrt{x'^2 + y'^2}\, dt \\ &= \sqrt{(3\sin t)^2 + (3\cos t)^2}\, dt \\ &= \sqrt{9}\, dt = 3\,dt \end{aligned} \right)$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3^4 \sin^2 t \cos t\, dt$$

$$= 3^4 \cdot \left[\frac{\sin^3 t}{3}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \boxed{3^3 \cdot 2}$$



C:

$\vec{r}(t) = \langle x(t), y(t)\rangle$, $a \le t \le b$

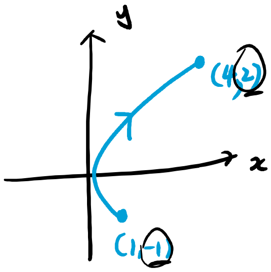
$\vec{r}'(t) = \langle x'(t), y'(t)\rangle$

$|\vec{r}'(t)| = \sqrt{x'^2 + y'^2}$

$$ds = \sqrt{(dx)^2 + (dy)^2}$$
$$= \sqrt{\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)(dt)^2}$$
$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$
$$= \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

$ds = |\vec{r}'(t)|dt$, $|\vec{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$

Problem 3. Evaluate $\int_C 2y\,ds$, where C is the arc of the curve $x = y^2$ from $(1, -1)$ to $(4, 2)$.



$C: \vec{r}(y) = \langle y^2, y \rangle$

$\vec{r}: \begin{cases} x = y^2 \\ y = y, \end{cases} \quad -1 \le y \le 2$

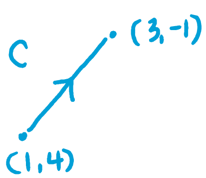
$ds = \sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2}\,dy = \sqrt{(2y)^2 + 1^2}\,dy = \sqrt{4y^2+1}\,dy$

$\int_C 2y\,ds = \int_{-1}^{2} 2y\underbrace{\sqrt{4y^2+1}}_{u}\,dy = \left[\frac{1}{4}\cdot\frac{2}{3}(4y^2+1)^{\frac{3}{2}}\right]_{-1}^{2}$

$du = 8y\,dy$

$= \boxed{\frac{1}{6}\left(17^{\frac{3}{2}} - 5^{\frac{3}{2}}\right)}$

Problem 4. Evaluate $\int_C (x^2 + y)\,ds$ where C consists of the line segment from the point $(1, 4)$ to $(3, -1)$.



$\vec{r}(t) = (1-t)\langle 1, 4\rangle + t\langle 3, -1\rangle$

$= \langle 1+2t, 4-5t \rangle,$

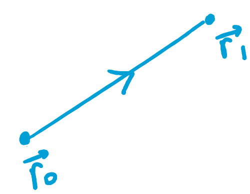
$0 \le t \le 1$

$\vec{r}'(t) = \langle 2, -5 \rangle$

$|\vec{r}'(t)| = \sqrt{2^2 + 5^2} = \sqrt{29}$

$\int_C x^2 + y\,\underbrace{ds}_{|\vec{r}'(t)|\,dt} = \int_0^1 \left((1+2t)^2 + (4-5t)\right)\sqrt{29}\,dt$

$= \boxed{\;\cdots\;}$



$C: \vec{r}(t) = (1-t)\vec{r}_0 + t\,\vec{r}_1,$

$0 \le t \le 1$

$\vec{r}(t) = \langle x(t), y(t) \rangle$

$ds = |\vec{r}'(t)|\,dt = \sqrt{x'^2 + y'^2}\,dt$

Problem 5. Evaluate $\int_C xyz\,ds$, where C is the line segment from the point $(-2, 0, 3)$ to $(0, 1, 2)$.

$C: \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$= (1-t)\langle -2, 0, 3\rangle + t\langle 0, 1, 2\rangle$

$= \langle -2+2t, t, 3-t \rangle,$

$0 \le t \le 1$

$|\vec{r}'(t)| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$

$\int_C xyz\,ds = \int_0^1 (-2+2t)\,t\,(3-t)\sqrt{6}\,dt$

$= \boxed{\;\cdots\;}$

Problem 6. Evaluate $\int_C y\,dx + x^2\,dy$, where C is decribed by $\mathbf{r}(t) = \langle \underbrace{3e^t}_{x}, \underbrace{e^{2t}}_{y} \rangle$, $0 \le t \le 1$.

$\vec{r}(t) = \langle x(t), y(t) \rangle$

$x = x(t)$ $\quad dx = x'(t)\,dt$

$y = y(t)$ $\quad dy = y'(t)\,dt$

$$\begin{cases} dx = 3e^t\,dt \\ dy = 2e^{2t}\,dt \end{cases}$$

$$= \int_0^1 e^{2t}\, 3e^t\,dt + (3e^t)^2 (2e^{2t}\,dt)$$

$$= \int_0^1 3e^{3t} + 18e^{4t}\,dt = \boxed{\ldots}$$

Problem 7. Evaluate $\int_C x\,dx + y\,dy$, where C is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(3, 1)$.

$$\vec{r}(y) = \begin{cases} x = 4 - y^2 \\ y = y \end{cases}, \quad -3 \le y \le 1$$

$$= \int_{-3}^1 (4 - y^2)(-2y\,dy) + y\,dy = \int_{-3}^1 -8y + 2y^3 + y\,dy$$

$$= \int_{-3}^1 2y^3 - 7y\,dy = \boxed{\ldots}$$

Problem 8. Evaluate $\int_C (x + y)dz + (y - x)dy + z\,dx$ where C: $x = t^4$, $y = t^3$, $z = t^2$, $0 \le t \le 1$.

$dx = 4t^3\,dt$ $\quad dy = 3t^2\,dt$ $\quad dz = 2t\,dt$

$$= \int_0^1 (t^4 + t^3)(2t\,dt) + (t^3 - t^4)\,3t^2\,dt + t^2(4t^3\,dt)$$

$$= \int_0^1 2t^5 + 2t^4 + 3t^5 - 3t^6 + 4t^5\,dt = \int_0^1 -3t^6 + 9t^5 + 2t^4\,dt = \boxed{\ldots}$$

$\int_C \vec{F}\cdot d\vec{r}$ $\quad \vec{F}=\langle xy, x^2, z\rangle \quad d\vec{r}=\langle dx, dy, dz\rangle$

Problem 9. Evaluate $\int_C xy\,dx + x^2\,dy + z\,dz$ where C is the line segment from $(0,-1,1)$ to $(2,3,-1)$.

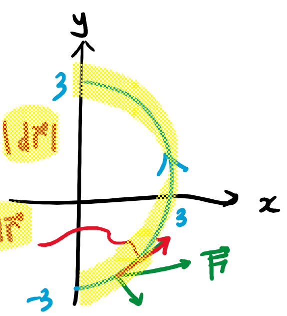
$C = (1-t)\vec{r}_0 + t\,\vec{r}_1, \quad 0\le t\le 1$

$$\vec{r}(t) = (1-t)\langle 0,-1,1\rangle + t\langle 2,3,-1\rangle$$
$$= \langle \underbrace{2t}_{x}, \underbrace{-1+4t}_{y}, \underbrace{1-2t}_{z}\rangle, \quad 0\le t\le 1$$

$$\begin{cases} dx = 2\,dt \\ dy = 4\,dt \\ dz = -2\,dt \end{cases}$$

$$\int_C \cdots = \int_0^1 (2t)(-1+4t)(2\,dt) + (2t)^2(4\,dt) + (1-2t)(-2\,dt)$$
$$= \boxed{\cdots}$$

Problem 10. Find the work done by the force field $\mathbf{F}(x,y) = \langle x^2, xy\rangle$ in moving an object counterclockwise around the right half of the circle $x^2+y^2=9$.



$|\vec{F}|\cos\theta\,|d\vec{r}|$ = $\vec{F}\cdot d\vec{r}$

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 0\,dt = \boxed{0}$$

$$\vec{F} = \langle x^2, xy\rangle \qquad d\vec{r} = \langle dx, dy\rangle = \langle -3\sin t, 3\cos t\rangle\,dt$$
$$\vec{F}\cdot d\vec{r} = \langle (3\cos t)^2, (3\cos t)(3\sin t)\rangle \cdot \langle -3\sin t, 3\cos t\rangle\,dt$$
$$= (-27\cos^2 t\sin t + 27\cos^2 t \sin t)\,dt$$
$$= 0\,dt$$

$$\vec{r}(t) = \begin{cases} x = 3\cos t \\ y = 3\sin t \end{cases}, \quad -\frac{\pi}{2}\le t\le\frac{\pi}{2}$$

$$\boxed{W = \int_C \underbrace{\vec{F}}_{\langle P,Q\rangle}\cdot\underbrace{d\vec{r}}_{\langle dx,dy\rangle} = \int_C P\,dx + Q\,dy}$$

$\vec{F} = \langle x, y\rangle$

Problem 11. Suppose we are moving a particle from the point $(0,0)$ to the point $(2,4)$ in a force field $\mathbf{F}(x,y) = \langle y^2, x\rangle$. Find $\int_C \mathbf{F}\cdot d\mathbf{r}$ where:

$$\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy, \qquad \vec{F} = \langle P, Q\rangle, \quad d\vec{r} = \langle dx, dy\rangle$$

(a) The particle travels along the line segment from $(0,0)$ to $(2,4)$.

$$\vec{r}(t) = (1-t)\langle 0,0\rangle + t\langle 2,4\rangle = \langle \underbrace{2t}_{x}, \underbrace{4t}_{y}\rangle, \quad 0 \le t \le 1$$

$$\vec{F}\cdot d\vec{r} = \langle y^2, x\rangle \cdot \langle 2\,dt, 4\,dt\rangle = (2y^2 + 4x)\,dt = (2(4t)^2 + 4(2t))\,dt$$

$$= (32t^2 + 8t)\,dt$$

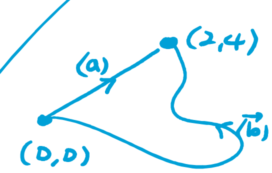
$$W = \int_0^1 32t^2 + 8t\,dt = \left[32\cdot\frac{t^3}{3} + 8\cdot\frac{t^2}{2}\right]_0^1 = \frac{32}{3} + \frac{8}{2} = \frac{32}{3} + 4 = \boxed{\frac{44}{3}}$$

(b) The particle travels along the curve $y = x^2$ from $(0,0)$ to $(2,4)$.

$$\vec{r}(x) = \begin{cases} x = x \\ y = x^2 \end{cases}, \quad 0 \le x \le 2$$

$$\vec{F}\cdot d\vec{r} = \langle y^2, x\rangle \cdot \langle dx, 2x\,dx\rangle = \overset{x^2}{y}{}^2\,dx + 2x^2\,dx = (x^4 + 2x^2)\,dx$$

$$W = \int_0^2 x^4 + 2x^2\,dx = \left[\frac{x^5}{5} + \frac{2}{3}x^3\right]_0^2 = \frac{32}{5} + \frac{16}{3} = \boxed{\frac{176}{15}}$$



different!

Problem 12. Find $\int_C \mathbf{F}\cdot d\mathbf{r}$, $C : \mathbf{r}(\mathbf{t}) = \langle \underbrace{t}_{x}, \underbrace{t^2}_{y}, \underbrace{t^4}_{z}\rangle$, $0 \le t \le 1$, and $\mathbf{F}(x,y,z) = \langle x, z^2, -4y\rangle$.

$$\vec{F}\cdot d\vec{r} = \langle t, (t^4)^2, (-4)t^2\rangle \cdot \langle dt, 2t\,dt, 4t^3\,dt\rangle = t\,dt + 2t^9\,dt + (-16)t^5\,dt$$

$$= (2t^9 - 16t^5 + t)\,dt$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 2t^9 - 16t^5 + t\,dt = \boxed{\dots}$$

Problem 13. Let $f(x, y) = \underbrace{3x}_{} + \cancel{x^2 y} - \cancel{yx^2}$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \nabla f$ and C is the curve given by $\mathbf{r}(t) = \langle 2t, t^2 \rangle$, $1 \le t \le 2$.

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(2)) - f(\vec{r}(1))$$

$$(\vec{r}(1) = \langle 2, 1 \rangle \quad \vec{r}(2) = \langle 4, 4 \rangle)$$

$$= 3 \cdot 4 - 3 \cdot 2$$

$$= 12 - 6 = \boxed{6}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r}$$

$$= \int_C \langle f_x, f_y \rangle \cdot \langle dx, dy \rangle$$

$$= \int_C \underbrace{\left(f_x \frac{dx}{dt} + f_y \frac{dy}{dt} \right)}_{\frac{df}{dt}} dt$$

$$= \int_C \frac{df}{dt} dt$$

$$= f$$

$f(x,y) \Rightarrow \nabla f = \langle f_x, f_y \rangle = \vec{F}$

$\Rightarrow \vec{F}$ is "conservative" if $\vec{F} = \nabla f$ for some f.

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = [f]_{\vec{r}(a)}^{\vec{r}(b)} = f(\vec{r}(b)) - f(\vec{r}(a))}$$

$C: \vec{r}(t)$, $a \le t \le b$

Problem 14. (a) Is $\mathbf{F}(x, y) = \langle \underbrace{3x^2 - 4y}_{P}, \underbrace{4y^2 - 2x}_{Q} \rangle$ a conservative vector field? If so, find a function f so that $\mathbf{F} = \nabla f$.

$$\left.\begin{array}{l} P_y = -4 \\ Q_x = -2 \end{array}\right\} \text{ different! } P_y \ne Q_x$$

$\Rightarrow \vec{F}$ is not conservative! (gradient)

$$\boxed{\vec{F} = \langle P, Q \rangle = \nabla f = \langle f_x, f_y \rangle}$$

$$\boxed{\left.\begin{array}{l} P = f_x \\ Q = f_y \end{array}\right\}} \Rightarrow P_y = f_{xy} = f_{yx} = Q_x$$

$$\Leftrightarrow \quad \boxed{P_y = Q_x}$$

f can be found by these!

(b) Is $\mathbf{F}(x, y) = \langle \underbrace{2x + 4y}_{P}, \underbrace{4x - 1}_{Q} \rangle$ a conservative vector field? If so, find a potential function for $\mathbf{F}$.

$$\left.\begin{array}{l} P_y = 4 \\ Q_x = 4 \end{array}\right) \text{ '=' } \vec{F} \text{ is conservative!}$$

$$\left.\begin{array}{l} P = f_x \\ Q = f_y \end{array}\right\} \longrightarrow f$$

$$\left\{\begin{array}{l} \boxed{f_x} = 2x + 4y \Rightarrow \boxed{\int f_x \, dx} = \int 2x + 4y \, dx = \boxed{x^2 + 4xy + g_1(y)}^{\,f} \\ \boxed{f_y} = 4x - 1 \Rightarrow \boxed{\int f_y \, dy} = \int 4x - 1 \, dy = \boxed{4xy - y + g_2(x)}^{\,f} \end{array}\right.$$

$$\boxed{f = x^2 + 4xy - y + C}$$

$$\nabla f = \langle f_x, f_y \rangle = \langle 2x + 4y, 4x - 1 \rangle$$

$P \quad Q \qquad P_y \overset{?}{=} Q_x$

Problem 15. Given $\mathbf{F}(x,y) = \langle 2xy^3, 3x^2y^2\rangle$. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the curve given by $\mathbf{r}(t) = \langle t^3+2t^2-t, 3t^4-t^2\rangle, 0 \le t \le 2$.

$$\left.\begin{array}{l} P_y = 6xy^2 \\ Q_x = 6xy^2 \end{array}\right) \text{`='} \Rightarrow \vec{F} \text{ conservative!}$$

$$\left.\begin{array}{l} f_x = P \\ f_y = Q \end{array}\right\} \to f$$

$$\left.\begin{array}{l} f_x = 2xy^3 \Rightarrow f = \int f_x dx = \int 2xy^3 dx = \boxed{x^2y^3 + g_1(y)} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\ f_y = 3x^2y^2 \Rightarrow f = \int f_y dy = \int 3x^2y^2 dy = \boxed{x^2y^3 + g_2(x)} \end{array}\right\} \Rightarrow f = x^2y^3 + C$$

$$\int_C \vec{F}\cdot d\vec{r} = f(\vec{r}(2)) - f(\vec{r}(0)) = (14)^2(44)^3 - 0 = \boxed{(14)^2(44)^3}$$

$\vec{r}(0) = \langle 0, 0\rangle$

$\vec{r}(2) = \langle 2^3 + 2\cdot 2^2 - 2, 3\cdot 2^4 - 2^2\rangle = \langle 14, 44\rangle$

$P \quad Q \quad R$

Problem 16. Given that $\mathbf{F} = \langle 4xe^z, \cos(y), 2x^2e^z\rangle$ is conservative and $\mathbf{r}(t) = \langle \sin(t), t, \cos(t)\rangle$, compute $\int_C \mathbf{F}\cdot d\mathbf{r}$ for $0 \le t \le \frac{\pi}{2}$. Note: We had to tell you $\mathbf{F}$ is conservative since we have not yet learned the testing criteria for conservativness in $\mathbb{R}^3$.

$$\boxed{\vec{F} = \langle P, Q, R\rangle = \nabla f = \langle f_x, f_y, f_z\rangle \qquad \left.\begin{array}{l} f_x = P \\ f_y = Q \\ f_z = R \end{array}\right\} \Rightarrow f}$$

$$\begin{array}{l} \qquad\qquad\qquad\qquad\qquad\qquad\qquad f \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\ f_x = 4xe^z \Rightarrow \int 4xe^z dx = \boxed{2x^2e^z + g_1(y,z)} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\ f_y = \cos(y) \Rightarrow \int \cos(y) dy = \boxed{\sin(y) + g_2(x,z)} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\ f_z = 2x^2e^z \Rightarrow \int 2x^2e^z dz = \boxed{2x^2e^z + g_3(x,y)} \end{array}$$

$$\boxed{f = 2x^2e^z + \sin(y) + C}$$

$$\left.\begin{array}{l} \vec{r}(\frac{\pi}{2}) = \langle \sin(\frac{\pi}{2}), \frac{\pi}{2}, \cos(\frac{\pi}{2})\rangle = \langle 1, \frac{\pi}{2}, 0\rangle \\ \vec{r}(0) = \langle \sin(0), 0, \cos(0)\rangle = \langle 0, 0, 1\rangle \end{array}\right\} \Rightarrow \int_C \vec{F}\cdot d\vec{r} = f(1, \tfrac{\pi}{2}, 0) - f(0,0,1)$$

$$= \left(2 + \sin(\tfrac{\pi}{2})\right) - 0 = \boxed{2 + \sin(\tfrac{\pi}{2})}$$