Note ♯5: Sections 15.1-15.3

Problem 1. (a) Evaluate $\int_{-3}^{3}\left(\int_{0}^{\pi/2} (y + y^2\cos x)\, dx\right) dy$. — iterated integral

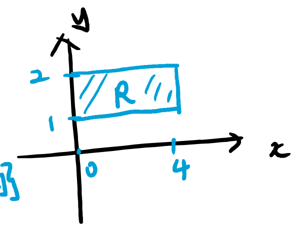
$$\int_0^{\frac{\pi}{2}} y + y^2\cos x \; dx = \left[yx + y^2\sin x\right]_0^{\frac{\pi}{2}} = \left(y\cdot\tfrac{\pi}{2} + y^2\sin(\tfrac{\pi}{2})\right) - \left(y\cdot 0 + y^2\sin 0\right) = \tfrac{\pi}{2}y + y^2$$

$$= \int_{-3}^{3} \tfrac{\pi}{2}y + y^2\,dy = \left[\tfrac{y^3}{3}\right]_{y=3}^{3} = 9-(-9) = \boxed{18}$$

odd function
$\int_{-3}^{3} y\,dy = 0$

(b) Evaluate $\iint_R \frac{x}{y^2}\,dA$, where $R = [0,4]\times[1,2]$.

$= \{(x,y) \mid x\in[0,4],\ y\in[1,2]\}$



Fubini's theorem

$$= \int_1^2\int_0^4 \frac{x}{y^2}\,dx\,dy \left(= \int_0^4\int_1^2 \frac{x}{y^2}\,dy\,dx\right)$$

$\frac{x}{y^2} = x\cdot\frac{1}{y^2}$

$\int_c^d\int_a^b f(x)\,g(y)\,dx\,dy$
$= \left(\int_a^b f\,dx\right)\left(\int_c^d g\,dy\right)$

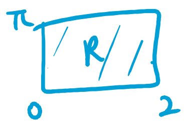
$$= \left(\int_0^4 x\,dx\right)\left(\int_1^2 \frac{1}{y^2}\,dy\right) = \left[\frac{x^2}{2}\right]_0^4\cdot\left[-y^{-1}\right]_1^2 = \frac{4^2}{2}\cdot\left(-\left(\frac{1}{2}-\frac{1}{1}\right)\right)$$

$\frac{1}{y^2} = y^{-2}$

$$= (8)\left(\tfrac{1}{2}\right) = \boxed{4}$$

Fubini's thm

(c) Evaluate $\int_0^2\int_0^\pi (y\cos(xy))dydx.$ $= \int_0^\pi \left(\int_0^2 y\cos(xy)\, dx \right) dy$



$$= \int_0^\pi \cancel{y}\left[\frac{\sin(xy)}{\cancel{y}}\right]_{x=0}^{2} dy$$

$$= \int_0^\pi \sin(2y) - 0 \; dy$$

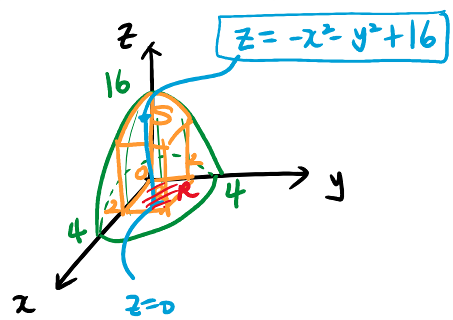
$$= -\left[\frac{\cos(2y)}{2}\right]_0^\pi$$

$$= -\frac{1}{2}\left(\cos(2\pi) - \cos(0)\right)$$

$$= 0.$$

Problem 2. Find the volume of the solid S that is bounded by the paraboloid $x^2 + y^2 + z = 16$, $z = 0$, $0 \le x \le \cancel{4}$, $0 \le y \le \cancel{4}$.

$2 \quad 2$



$$V = \int_0^2\int_0^2 (-x^2 - y^2 + 16) - 0 \; dx\,dy$$

$$= \int_0^2 \left[-\frac{x^3}{3} - y^2 x + 16x\right]_{x=0}^{2} dy$$

$$= \int_0^2 -\frac{8}{3} - 2y^2 + 32 \; dy = \int_0^2 -2y^2 + \frac{88}{3} dy = \left[-\frac{2}{3}y^3 + \frac{88}{3}y\right]_0^2 = -\frac{16}{3} + \frac{176}{3} = \boxed{\frac{160}{3}}$$

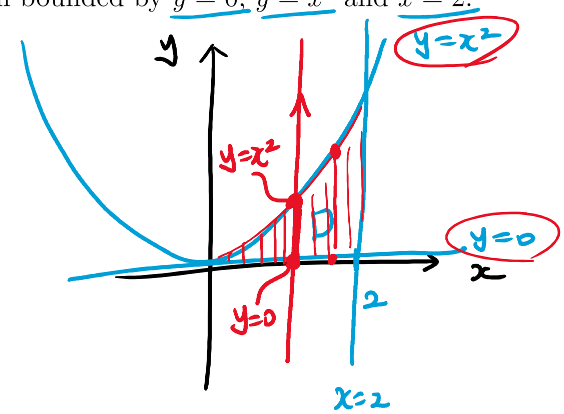
Problem 3. Evaluate $\int_1^4 \left(\int_1^{\sqrt{x}} (x+y)dy\right)dx.$ $= \int_1^4 \left[xy + \frac{y^2}{2}\right]_{y=1}^{\sqrt{x}} dx$

$$= \int_1^4 \left(x\sqrt{x} + \frac{x}{2}\right) - \left(x + \frac{1}{2}\right) dx = \int_1^4 x^{\frac{3}{2}} - \frac{x}{2} - \frac{1}{2} dx$$

$$= \left[\frac{2}{5}x^{\frac{5}{2}} - \frac{x^2}{4} - \frac{x}{2}\right]_1^4 = \cdots$$

Problem 4. Evaluate $\iint_D xe^y dA$, where D is the region bounded by $y = 0$, $y = x^2$ and $x = 2$.

$$= \int_0^2 \left(\int_0^{x^2} xe^y dy\right) dx$$



$$= \int_0^2 \left[xe^y\right]_{y=0}^{x^2} dx$$

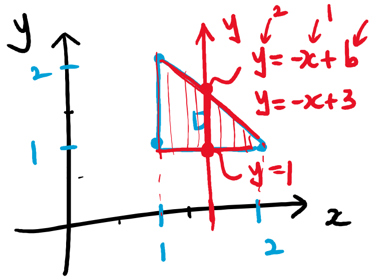
$$= \int_0^2 xe^{x^2} - x \, dx$$

$$= \left[\frac{1}{2}e^{x^2} - \frac{x^2}{2}\right]_0^2$$

$$= \left(\frac{1}{2}e^4 - \frac{4}{2}\right) - \left(\frac{1}{2}\right)$$

$$= \boxed{\frac{1}{2}e^4 - \frac{5}{2}}$$

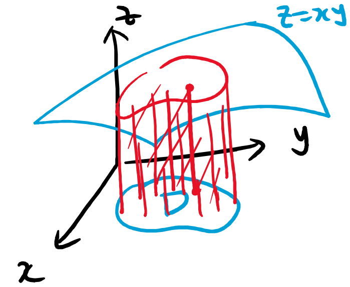
Problem 5. Find the volume of the solid under the surface $z = xy$ and above the triangle with vertices $(1, 1), (1, 2)$ and $(2, 1)$.



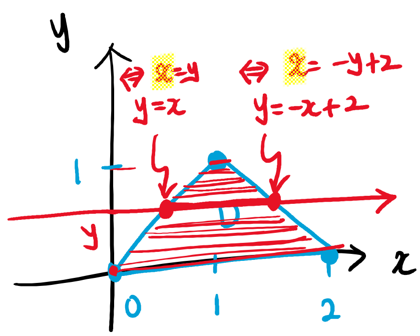
$$V = \iint_D z \, dA$$

$$= \int_1^2 \left(\int_1^{-x+3} xy \, dy \right) dx$$

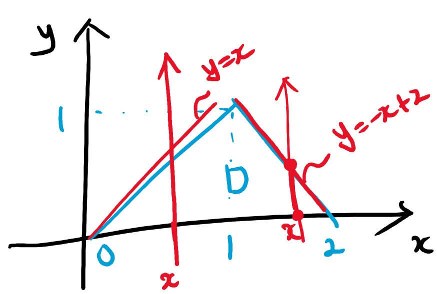
$$= \cdots$$



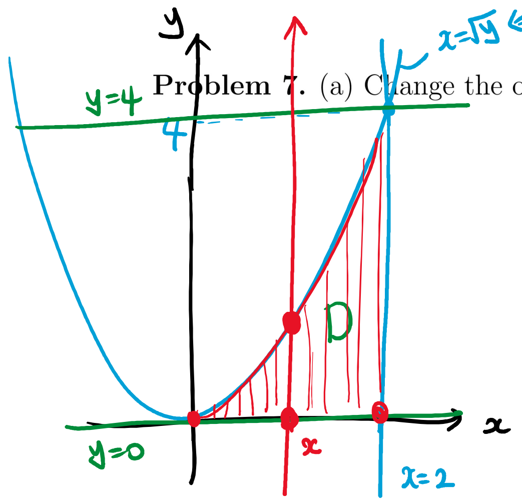
Problem 6. Set up but do not evaluate $\iint_D ye^x dA$ in two different iterated integrals, where D is the triangular region with vertices $(0, 0), (1, 1)$ and $(2, 0)$.



$$\boxed{\int_0^1 \int_y^{-y+2} y e^x \, dx \, dy}$$



$$\boxed{\int_0^1 \left(\int_0^x y e^x \, dy \right) dx + \int_1^2 \left(\int_0^{-x+2} y e^x \, dy \right) dx}$$



$x=\sqrt{y} \Leftrightarrow x^2=y$

Problem 7. (a) Change the order of integration in $\int_{0}^{4}\left(\int_{\sqrt{y}}^{2} f(x,y)\,dx\right)dy$.

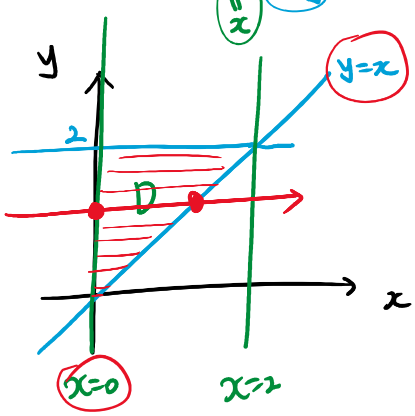
$y=4$, $y=0$, $2=x$, $\sqrt{y}=x$

$$\cancel{= \int_{\sqrt{y}}^{2}\int_{0}^{4} f\,dy\,dx}$$

$$\boxed{= \int_{0}^{2}\left(\int_{0}^{x^2} f(x,y)\,dy\right)dx}$$

(b) Evaluate $\int_0^2 \int_x^2 e^{-y^2}\,dy\,dx = \int_0^2\left(\int_0^y e^{-y^2}\,dx\right)dy$

$2=x$, $0=x$, $2=y$, $x=y$

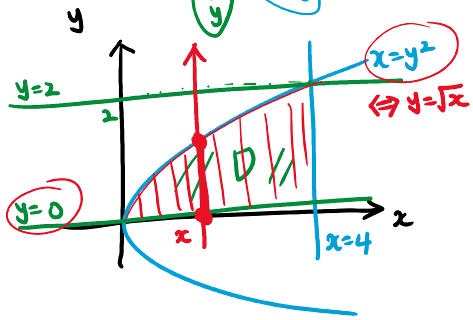


$$= \int_0^2 e^{-y^2}\left[x\right]_0^y\,dy = \int_0^2 e^{-y^2}\cdot y\,dy$$

$$= \left[-\frac{1}{2}e^{-y^2}\right]_0^2 = \boxed{-\frac{1}{2}\left(e^{-4}-1\right)}$$

(c) Evaluate $\int_0^2 \int_{y^2}^4 \sqrt{x} \sin x \, dx dy = \int_0^4 \int_0^{\sqrt{x}} \sqrt{x} \sin x \, dy \, dx$

$0 = y$, $2 = y$, $y^2 = x$, $4 = x$



$$= \int_0^4 \underbrace{x}_{f} \underbrace{\sin x}_{g'} \, dx$$

$\int f g' = f g - \int f' g$ Integration by Parts

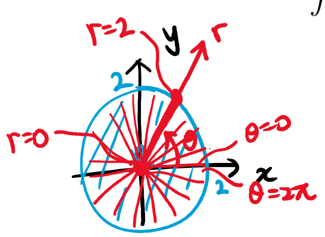
$$= \Big[\underbrace{x}_{f}\underbrace{(-\cos x)}_{g}\Big]_0^4 - \int_0^4 \underbrace{1}_{f'}\underbrace{(-\cos x)}_{g} dx$$

$$= -4\cos 4 + \int_0^4 \cos x \, dx$$

$$= \boxed{-4\cos(4) + \sin(4)}$$

$x = r\cos\theta \quad y = r\sin\theta \quad x^2+y^2 = r^2 \quad dA = dx\,dy = r\,dr\,d\theta$

Problem 8. Evaluate $\iint_R (x+2) dA$, where R is the region bounded by the circle $x^2 + y^2 = 4$.



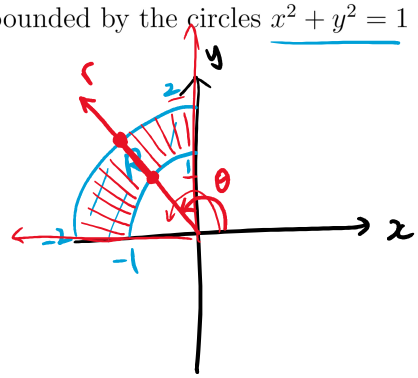
$$= \int_0^{2\pi} \int_0^2 (r\cos\theta + 2) \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 r^2\cos\theta + 2r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^3}{3}\cos\theta + r^2\right]_{r=0}^{2} d\theta = \int_0^{2\pi} \cancel{\frac{8}{3}\cos\theta} + 4 \, d\theta$$

$$= \left[4\theta\right]_0^{2\pi} = 8\pi - 0 = \boxed{8\pi}$$

Problem 9. Set up but do not evaluate $\iint_R 4y\,dA$, where R is the region in the second quadrant bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.

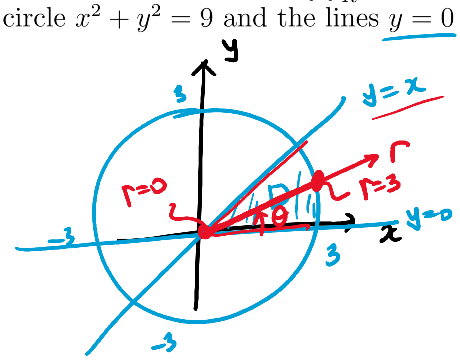


$$\iint_R 4y\,dA = \boxed{\int_{\frac{\pi}{2}}^{\pi}\int_1^2 4\,r\sin\theta\,(r\,dr\,d\theta)}$$

$y=\sqrt{3}x \Rightarrow \theta=\frac{\pi}{3}$

$y=\frac{1}{\sqrt{3}}x \Rightarrow \theta=\frac{\pi}{6}$

Problem 10. Evaluate $\iint_R 3x^2\,dA$, where R is the region in the first quadrant enclosed by the circle $x^2+y^2=9$ and the lines $y=0$ and $y=x$.



$$=\int_0^{\frac{\pi}{4}}\int_0^3 \underbrace{3(r\cos\theta)^2(r\,dr\,d\theta)}_{3r^3\cos^2\theta}$$

$$=3\underbrace{\int_0^3 r^3\,dr}_{\left[\frac{r^4}{4}\right]_0^3=\frac{3^4}{4}}\ \underbrace{\int_0^{\frac{\pi}{4}}\cos^2\theta\,d\theta}_{}$$

$$\int_0^{\frac{\pi}{4}}\cos^2\theta\,d\theta = \int_0^{\frac{\pi}{4}}\frac{\cos(2\theta)+1}{2}\,d\theta = \left[\frac{1}{2}\frac{\sin(2\theta)}{2}+\frac{1}{2}\theta\right]_0^{\frac{\pi}{4}} = \left(\frac{1}{4}+\frac{\pi}{8}\right)$$

$$=\boxed{\frac{3^5}{4}\left(\frac{1}{4}+\frac{\pi}{8}\right)}$$

Power Reduction formulas

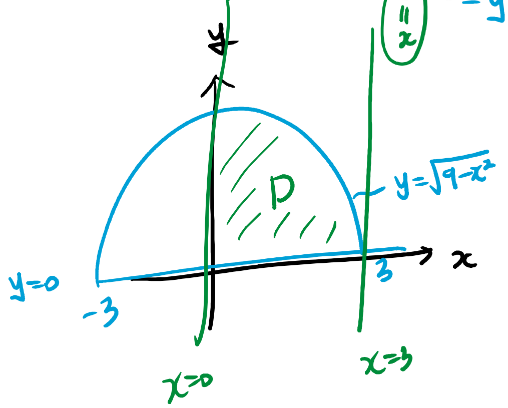
$$\cos^2\theta=\frac{1+\cos(2\theta)}{2}$$

$$\sin^2\theta=\frac{1-\cos(2\theta)}{2}$$

Problem 11. (a) Change $\int_0^3\int_0^{\sqrt{9-x^2}} x^2\,dy\,dx$ to polar coordinates. Do not evaluate the integral.

$9-x^2=y^2$

$\Leftrightarrow 9=x^2+y^2$



$$\boxed{=\int_0^{\frac{\pi}{2}}\int_0^3 (r\cos\theta)^2\,(r\,dr\,d\theta)}$$

(b) Change $\int_0^4\int_0^{\sqrt{4x-x^2}}\sqrt{x^2+y^2}\,dy\,dx$ to polar coordinates. Do not evaluate the integral.

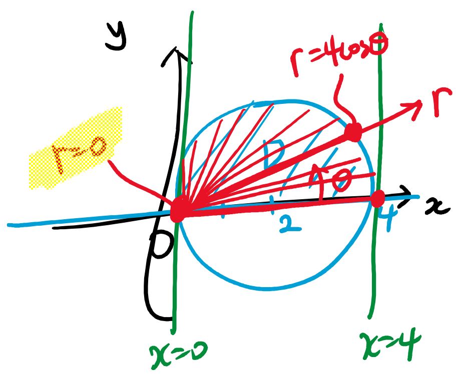
$y^2=4x-x^2$

$\Leftrightarrow \boxed{x^2-4x+y^2=0}$

$\Leftrightarrow (x-2)^2-4+y^2=0$

$\Leftrightarrow (x-2)^2+y^2=4$

$(2,0)$ $r=2$



$$\boxed{\int_0^{\frac{\pi}{2}}\int_0^{4\cos\theta} r\,(r\,dr\,d\theta)}$$

$r^2-4r\cos\theta=0$

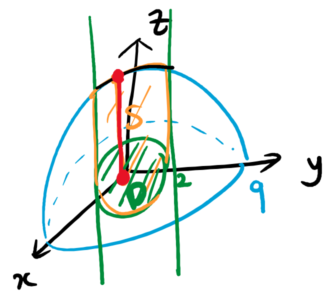
$\Leftrightarrow r(r-4\cos\theta)=0$

$\Leftrightarrow r=0$ or $r=4\cos\theta$

Problem 12. Set up but do not evaluate an integral that gives the volume of the solid that lies above the xy-plane, below the sphere $x^2 + y^2 + z^2 = 81$ and inside the cylinder $x^2 + y^2 = 4$ in polar coordinates.

$z^2 = 81 - x^2 - y^2$

$z = \sqrt{81 - x^2 - y^2}$

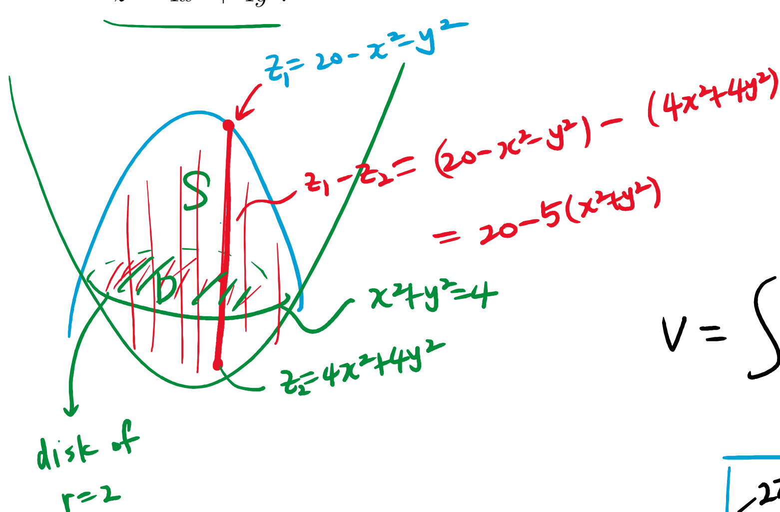


$$\iint_D \sqrt{81 - x^2 - y^2}\, dA = \int_0^{2\pi}\left(\int_0^2 \sqrt{81 - r^2}\; r\,dr\right)d\theta$$

$$= \left(\int_0^2 \sqrt{81 - r^2}\; r\,dr\right)\underbrace{\left(\int_0^{2\pi} d\theta\right)}_{2\pi}$$

$$\left[-\frac{1}{2}\frac{2}{3}(81 - r^2)^{\frac{3}{2}}\right]_0^2$$

Problem 13. Find the volume of the solid bounded by the paraboloids $z = 20 - x^2 - y^2$ and $z = 4x^2 + 4y^2$.



$20 - x^2 - y^2 = 4x^2 + 4y^2$

$20 = 5x^2 + 5y^2$

$4 = x^2 + y^2 = r^2$

$$V = \iint_D 20 - 5(x^2 + y^2)\, dA$$

$$= \boxed{\int_0^{2\pi}\int_0^2 (20 - 5r^2)(r\,dr\,d\theta)}$$

$= \ldots$