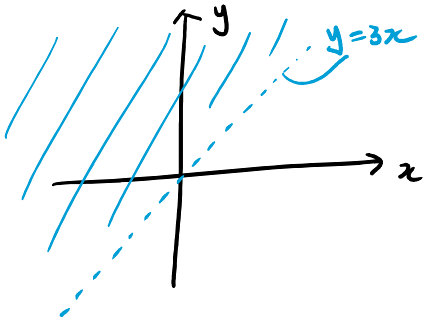
NOTE #3: SECTIONS 14.1-14.5

Problem 1. Find and sketch the domains of the following functions.

a) $f(x, y) = \ln(\underbrace{y - 3x}_{>0})$

$y - 3x > 0 \Leftrightarrow y > 3x$

$D = \{(x, y) \mid y > 3x\}$ $y = 3x$

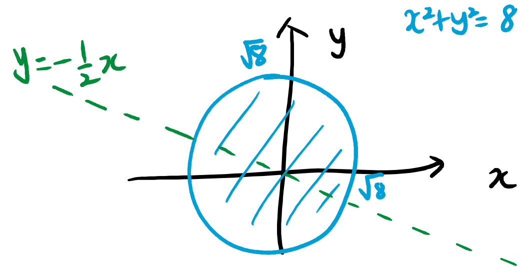


b) $f(x, y) = \dfrac{\sqrt{8 - x^2 - y^2}}{\underbrace{x + 2y}_{\neq 0}} \geq 0$

$8 - x^2 - y^2 \geq 0 \Leftrightarrow 8 \geq x^2 + y^2$

$x + 2y \neq 0 \Leftrightarrow 2y \neq -x \Leftrightarrow y \neq -\frac{1}{2}x$

$D = \{(x, y) \mid x^2 + y^2 \leq 8,\ y \neq -\frac{1}{2}x\}$



Problem 2. Sketch several level curves for the following surfaces: $z = k$ $z = 1, 2, \cdots$

a) $f(x, y) = 2 + 4x - y = z = k$

$2 + 4x - y = k$

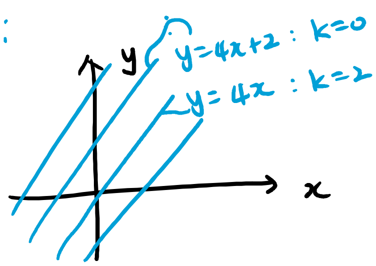
$\Leftrightarrow 4x + 2 - k = y$

$\Leftrightarrow \boxed{y = 4x + 2 - k}$

$k = 0$: $y = 4x + 2$

$k = 1$: $y = 4x + 2 - 1 = 4x + 1$

$\vdots$



b) $f(x, y) = \sqrt{9 - x^2 - y^2} = z = k$

$\Leftrightarrow 9 - x^2 - y^2 = k^2$

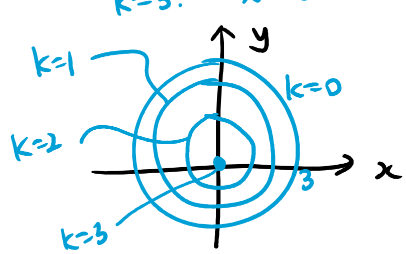
$\Leftrightarrow x^2 + y^2 = 9 - k^2$

$k = 0$: $x^2 + y^2 = 9$

$k = 1$: $x^2 + y^2 = 8$

$k = 2$: $x^2 + y^2 = 5$

$k = 3$: $x^2 + y^2 = 0$



Problem 3. a) Describe the level surfaces of $f(x, y, z) = x + y + z = w = k$

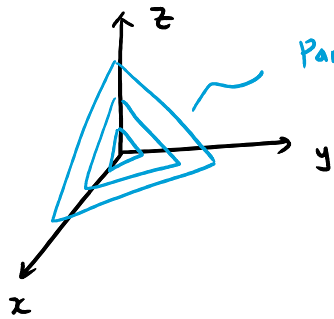
$x + y + z = k$

$k=0$: $x+y+z=0$ normal $\langle 1,1,1 \rangle$

$k=1$: $x+y+z=1$ "

$k=2$: $x+y+z=2$ "

$\vdots$ $\vdots$



Parallel planes

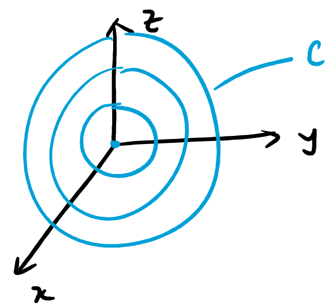
b) Describe the level surfaces of $f(x, y, z) = x^2 + y^2 + z^2 = w = k$ - constant

$k=0$ $x^2+y^2+z^2=0$

$k=1$ $x^2+y^2+z^2=1$

$k=2$ $\vdots$

$\vdots$



concentric spheres.

Problem 4. a) Find $f_x(-1, 2)$ and $f_y(-1, 2)$ for $f(x, y) = x^3 - y^4 - 6x^2y^3$.

"all other variables are constant"

$f_x(x,y) = 3x^2 - 0 - 6(2x)y^3 = 3x^2 - 12xy^3$

$\boxed{f_x(-1,2)} = 3(-1)^2 - 12(-1)(2^3) = 3 + 96 = \boxed{99}$

$f_y = 0 - 4y^3 - 6x^2(3y^2) = -4y^3 - 18x^2y^2$

$\boxed{f_y(-1,2)} = -4(2^3) - 18(-1)^2(2^2) = -32 - 72 = \boxed{-104}$

b) Find $f_x(x, y)$ and $f_y(x, y)$ for $f(x, y) = x^2 e^{\cos(2x^4y^2)}$.

$f = gh$
$f' = g'h + gh'$

$(f(g(x)))' = f'(g(x)) \cdot g'(x)$

$f_x = (2x)e^{\cos(2x^4y^2)} + x^2 e^{\cos(2x^4y^2)} \cdot (-\sin(2x^4y^2)) \cdot (8x^3y^2)$

$= \boxed{(2x - 8x^5y^2 \sin(2x^4y^2))\, e^{\cos(2x^4y^2)}}$

$f_y = x^2 e^{\cos(2x^4y^2)} \cdot (-\sin(2x^4y^2))(4x^4y)$

$= \boxed{-4x^6 y \sin(2x^4y^2)\, e^{\cos(2x^4y^2)}}$

Problem 5. Find all second order partial derivatives for $f(x, y) = \ln(2x + 3y)$.

f_{xx}, $f_{xy} = f_{yx}$, f_{yy}

- $f_x = \dfrac{2}{2x+3y}$
- $f_y = \dfrac{3}{2x+3y}$

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\left(\frac{2}{2x+3y}\right) = \frac{\partial}{\partial x}\left(2(2x+3y)^{-1}\right) = 2\cdot(-1)(2x+3y)^{-2}\cdot 2 = \boxed{-\frac{4}{(2x+3y)^2}}$$

$$f_{xy} = f_{yx} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\left(\frac{2}{2x+3y}\right) = -\frac{2\cdot 3}{(2x+3y)^2} = \boxed{-\frac{6}{(2x+3y)^2}}$$

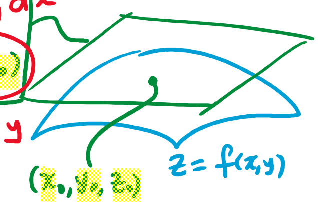
$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{3}{2x+3y}\right) = \boxed{-\frac{9}{(2x+3y)^2}}$$

Problem 6. Find the equation of the tangent plane to the surface $z = e^{x-y}$ at the point $(2, 2, 1)$.

$$f_x = e^{x-y}\Big|_{(2,2)} = e^{2-2} = e^0 = \boxed{1}$$

$$f_y = -e^{x-y}\Big|_{(2,2)} = -e^{2-2} = -e^0 = \boxed{-1}$$

$$\boxed{\underbrace{z - z_0}_{dz} = f_x(x_0,y_0)\underbrace{(x-x_0)}_{dx} + f_y(x_0,y_0)\underbrace{(y-y_0)}_{dy}}$$



$$z - 1 = (1)(x-2) + (-1)(y-2)$$

$$\Leftrightarrow \boxed{z - 1 = (x-2) - (y-2)}$$

$$\Leftrightarrow \boxed{0 = x - y - z + 1}$$

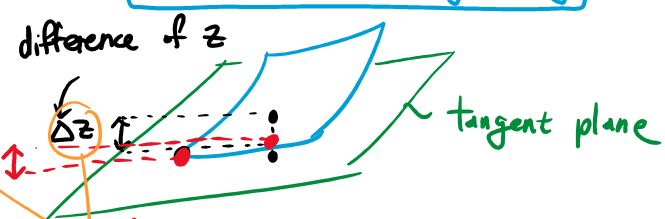
Problem 7. Find the differential of $z = e^{-2x} \sin(\pi y)$.

$$\boxed{dz = f_x(x,y)\,dx + f_y(x,y)\,dy}$$

$$f_x = -2e^{-2x}\sin(\pi y)$$

$$f_y = e^{-2x}(\cos(\pi y))(\pi)$$

$$\boxed{dz = (-2e^{-2x}\sin(\pi y))\,dx + (\pi e^{-2x}\cos(\pi y))\,dy}$$



differential of z

: difference of z values on the tangent plane

$$f(x,y) - f(x_0,y_0) = \Delta z \approx dz$$

$$\boxed{f(x,y) \approx f(x_0,y_0) + dz}$$

$$// \; 1-(1.02)(0.97)\cos(\pi(0.97))$$

Problem 8. Use differentials to approximate $f(1.02, 0.97)$ for $f(x, y) = 1 - xy\cos(\pi y)$.

$$f(1.02, 0.97) \approx f(1,1) + dz = 2 - 0.01 = \boxed{1.99}$$

$$f(x,y) - f(x_0,y_0) = \Delta z \approx dz$$

$$\boxed{f(x,y) \approx f(x_0,y_0) + dz}$$

$$\boxed{f(1,1)} = 1-(1)(1)\cos(\pi\cdot 1) = 1-\cos(\pi) = 1-(-1) = \boxed{2}$$

$$dz = f_x(1,1)\,dx + f_y(1,1)\,dy$$

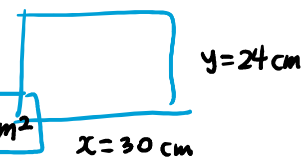
$$f_x = -y\cos(\pi y)\Big|_{(1,1)} = -\cos(\pi) = 1$$

$$f_y = -x\cos(\pi y) + (-xy)(-\sin(\pi y)\cdot\pi)\Big|_{(1,1)} = -\cos(\pi) + \pi\sin(\pi) = -(-1) + 0 = 1$$

$$\boxed{dz} = (1)(0.02) + (1)(-0.03) = 0.02 - 0.03 = \boxed{-0.01}$$

Problem 9. The length and width of a rectangle are measured as 30 cm and 24 cm, respectively, with an error in measurement of 0.1 cm in both. Use differentials to approximate the maximum error in the calculated area of the rectangle.

Let x = length, y = width. Then, the area $z = \underbrace{xy}_{f}$.



$$\Delta z \approx dz = f_x\,dx + f_y\,dy = (24)(0.1) + (30)(0.1) = 2.4 + 3 = \boxed{5.4\text{ cm}^2}$$

$$f_x = y\Big|_{y=24} = 24$$

$$f_y = x\Big|_{x=30} = 30$$

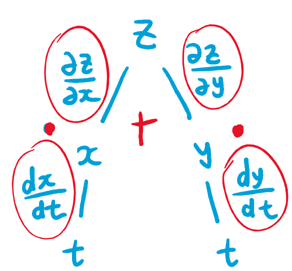
$$x = f(g(t)) \qquad \frac{dx}{dt} = f'(g(t))\,g'(t)$$

$$\sin(e^t)\cdot e^t$$

$$f = \cos(t)$$

$$g = e^t$$

Problem 10. a) If $z = \ln(9x - 6y)$, $x = \cos(e^t)$, $y = \sin^3(4t)$, find $\frac{dz}{dt}$.



$$\frac{\partial z}{\partial x} = \frac{9}{9x-6y} = \frac{3}{3x-2y}$$

$$\frac{\partial z}{\partial y} = \frac{-6}{9x-6y} = -\frac{2}{3x-2y}$$

$$\frac{dx}{dt} = -\sin(e^t)\cdot e^t$$

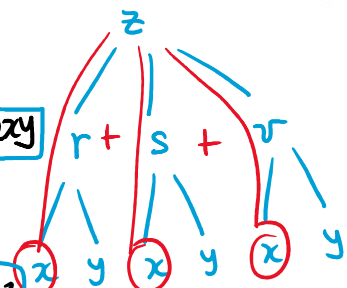
$$\frac{dy}{dt} = 3\sin^2(4t)\cdot\cos(4t)\cdot 4$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$\boxed{\frac{dz}{dt} = \left(\frac{3}{3x-2y}\right)\left(-e^t\sin(e^t)\right) + \left(-\frac{2}{3x-2y}\right)\left(12\sin^2(4t)\cos(4t)\right)}$$

b) If $z = r^3 + s + v^2, r = xe^y, s = ye^x, v = x^2y$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$$\frac{\partial z}{\partial x} = \underbrace{\frac{\partial z}{\partial r}}_{3r^2}\cdot\underbrace{\frac{\partial r}{\partial x}}_{e^y} + \underbrace{\frac{\partial z}{\partial s}}_{1}\cdot\underbrace{\frac{\partial s}{\partial x}}_{ye^x} + \underbrace{\frac{\partial z}{\partial v}}_{2v}\cdot\underbrace{\frac{\partial v}{\partial x}}_{2xy} = \boxed{3r^2\cdot e^y + 1\cdot ye^x + 2v\cdot 2xy}$$

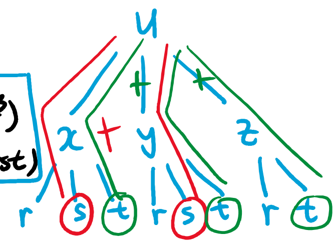


$$\frac{\partial z}{\partial y} = \underbrace{\frac{\partial z}{\partial r}}_{3r^2}\cdot\underbrace{\frac{\partial r}{\partial y}}_{xe^y} + \underbrace{\frac{\partial z}{\partial s}}_{1}\cdot\underbrace{\frac{\partial s}{\partial y}}_{e^x} + \underbrace{\frac{\partial z}{\partial v}}_{2v}\cdot\underbrace{\frac{\partial v}{\partial y}}_{x^2} = \boxed{3r^2 xe^y + 1\cdot e^x + 2v x^2}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial z}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

c) If $u = x^4y + y^2z^3, x = 2rs + 3st, y = rs^2t, z = r^2e^{2t}$, find $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$.

$$\frac{\partial u}{\partial s} = \underbrace{\frac{\partial u}{\partial x}}_{4x^3y}\cdot\underbrace{\frac{\partial x}{\partial s}}_{2r+3t} + \underbrace{\frac{\partial u}{\partial y}}_{x^4+2yz^3}\cdot\underbrace{\frac{\partial y}{\partial s}}_{2rst} + \frac{\partial u}{\partial z}\cdot 0 = \boxed{(4x^3y)(2r+3t) + (x^4+2yz^3)(2rst)}$$



$$\frac{\partial u}{\partial t} = \underbrace{\frac{\partial u}{\partial x}}_{4x^3y}\cdot\underbrace{\frac{\partial x}{\partial t}}_{3s} + \underbrace{\frac{\partial u}{\partial y}}_{x^4+2yz^3}\cdot\underbrace{\frac{\partial y}{\partial t}}_{rs^2} + \underbrace{\frac{\partial u}{\partial z}}_{3y^2z^2}\cdot\underbrace{\frac{\partial z}{\partial t}}_{2r^2e^{2t}} = \cdots$$

d) If $z = x^4 + xy^3, x = uv^3 + w^4, y = u + ve^w$, find find $\frac{\partial z}{\partial u}$ when $u = 1, v = 1, w = 0$.

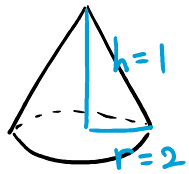
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = 12\cdot 1 + 12\cdot 1 = \boxed{24}$$

$$\begin{cases} \frac{\partial z}{\partial x} = 4x^3 + y^3 = 4\cdot 1^3 + 2^3 = 12 \\ \frac{\partial z}{\partial y} = 3xy^2 = 3\cdot 1\cdot 2^2 = 12 \\ \frac{\partial x}{\partial u} = v^3 = 1^3 = 1 \\ \frac{\partial y}{\partial u} = 1 \end{cases}$$

$x = (1)(1)^3 + 0^4 = 1$

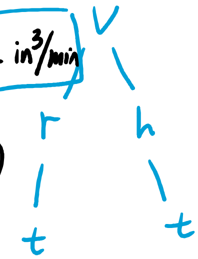
$y = 1 + 1\cdot e^0 = 1 + 1 = 2$

Problem 11. The height and radius of a right circular cone are changing with respect to time. If the base radius of the cone is increasing at a rate of $\frac{1}{4}$ inches per minute while the height is decreasing at a rate $\frac{1}{10}$ inches per minute, find the rate in which the volume of the cone is changing when the radius of the cone is 2 inches and the height of the cone is 1 inch.



Volume $V = \frac{1}{3}\pi r^2 h = f(r,h)$

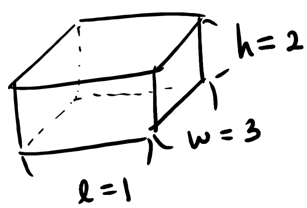
$\frac{4\pi}{3}\left(\frac{3}{20}\right) = \boxed{\frac{\pi}{5}\ in^3/min}$



$$\boxed{\frac{dV}{dt}} = \frac{\partial V}{\partial r}\cdot\frac{dr}{dt} + \frac{\partial V}{\partial h}\cdot\frac{dh}{dt} = \left(\frac{4\pi}{3}\right)\left(\frac{1}{4}\right) + \left(\frac{4\pi}{3}\right)\left(-\frac{1}{10}\right)$$

$$\begin{cases} \frac{dr}{dt} = \frac{1}{4}, \quad \frac{dh}{dt} = -\frac{1}{10} \\ \frac{\partial V}{\partial r} = \frac{2}{3}\pi r h \Big|_{r=2, h=1} = \frac{4\pi}{3}, \quad \frac{\partial V}{\partial h} = \frac{1}{3}\pi r^2 \Big|_{r=2} = \frac{4}{3}\pi \end{cases}$$

Problem 12. The length l, width w and height h of a box change with time. At a certain instant, the dimensions are $l = 1$ m, $w = 3$ m and $h = 2$ m, and l and w are increasing at rate of 2 m/s while h is decreasing at a rate of 3 m/s. At that same instant, find the rate at which the surface area is changing.



$\frac{dl}{dt} = \frac{dw}{dt} = 2, \quad \frac{dh}{dt} = -3$

$A = 2lw + 2wh + 2lh$

$$\frac{dA}{dt} = A_l \cdot \frac{dl}{dt} + A_w \cdot \frac{dw}{dt} + A_h \cdot \frac{dh}{dt} = (10)(2) + (6)(2) + (8)(-3)$$

$$= 20 + 12 - 24$$

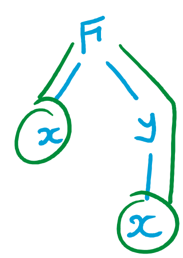
$$= \boxed{8\ m^2/s}$$

$$\begin{cases} A_l = 2w + 2h = 2\cdot 3 + 2\cdot 2 = 10 \\ A_w = 2l + 2h = 2\cdot 1 + 2\cdot 2 = 6 \\ A_h = 2w + 2l = 2\cdot 3 + 2\cdot 1 = 8, \end{cases}$$

Problem 13. Find dy/dx.

a) $y\cos x = x^2 + y^2$. $\Leftrightarrow \underbrace{x^2 + y^2 - y\cos x = 0}_{F(x,y)}$

$F(x,y) = 0 \Rightarrow \frac{dy}{dx} =$



$$\frac{dy}{dx} = \frac{-F_x}{F_y} = \boxed{-\frac{2x + y\sin x}{2y - \cos x}}$$

$$\begin{cases} F_x = 2x + y\sin x \\ F_y = 2y - \cos x \end{cases}$$

$$\frac{dF}{dx} = \boxed{\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx} = 0}$$

Implicit differentiation $\boxed{\frac{dy}{dx} = \frac{-\partial F/\partial x}{\partial F/\partial y}}$ ✓

b) $e^y \sin x = x + xy$ $\Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y}$

$$\underbrace{e^y \sin x - x - xy}_{F(x,y)} = 0$$

$$\begin{cases} F_x = e^y \cos x - 1 - y \\ F_y = e^y \sin x - x \end{cases} \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = \boxed{-\frac{e^y \cos(x) - 1 - y}{e^y \sin(x) - x}}$$

Problem 14. Find $\partial z/\partial x$ and $\partial z/\partial y$.

a) $x^2 + 2y^2 + 3z^2 = 1$

b) $yz + x \ln y = z^2$