Week in Review
Math 152

Week 14

10.1 Review of Parametric Equations

10.2 Arc Length and Surface Area of Parametric Curves

10.3 Polar Coordinates

Review of Parametric Equations

Finding Cartesian from Parametric Equations

Identify the particle's path by finding a Cartesian equation for it.

- $x = 4\sin t\,, y = 5\cos t\ (0 \le t \le 2\pi)$

From $\sin^2 x + \cos^2 x = 1$,

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$$

- $x = 1 + \sin t\,, y = \cos t - 1\ (0 \le t \le \pi)$

From $\sin^2 x + \cos^2 x = 1$,

$$(x-1)^2 + (y+1)^2 = 1$$

$$-1 \le x \le 1, -2 \le y \le -1$$

- $x = \sqrt{t+1}, y = \sqrt{t}\ (t \ge 0)$

By equating for t,

$$x^2 - 1 = y^2$$

$$1 \le x, 0 \le y$$

Parametric Equations $\Rightarrow$ Cartesian equation

(Remove the parameter)

A parametric Equation has an orientation and domain

Find a parametrization for the curve.

- The lower half of the parabola $x - 1 = y^2$

$y \leq 0$

$x = t^2 + 2t, y = t, t \leq 0$

- The ray (half line) with initial point (2, 3) that passes through the point (0,0)

$\frac{y}{x} = \frac{3-0}{2-0} \Rightarrow y = \frac{3}{2}x \ (0 \leq x)$

$x = t, y = \frac{3}{2}t \ \ (0 \leq t)$

- Find parametric equations and a parameter interval for the motion of a particle starting at the point (2, 0) and tracing the top half of the circle $x^2 + y^2 = 4$ four times.

$x(0) = 2, y(0) = 0$

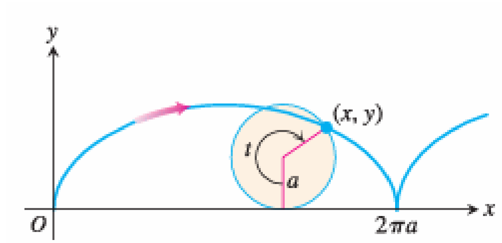
$x = 2\cos t\,, y = 2|\sin t|$

$0 \leq t \leq 4\pi$

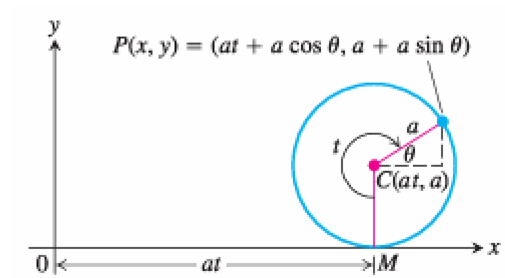
Parametric Equations are not unique

Review of Parametric Equations

A wheel of radius a rolls along a horizontal straight line. Find parametric equations for the path traced by a point P on the wheel's circumference. The path is called a cycloid.



$$x = at + a\cos t$$
$$y = a + a\sin t$$



Review of Parametric Equations

Find the velocity and speed of the motion along the following parameterized curve as a function of time : $x = t, y = t^2$

Position: $\mathbf{x}(t) = \big(x(t), y(t)\big)$
$\mathbf{x}(t) = (t, t^2)$

Velocity: $\mathbf{v}(t) = \mathbf{x}'(t) = \big(x'(t), y'(t)\big)$
$\mathbf{v}(t) = (1,2t)$

Speed : $\|\mathbf{v}(t)\| = \sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2}$
$\|\mathbf{v}(t)\| = \sqrt{1 + 4t^2}$

Length of Parametric Equations

Find the lengths of the curves

- $x = \cos t\,, y = t + \sin t \;\; (0 \le t \le \pi)$
- $x = \frac{1}{3}(2t+3)^{3/2}, y = t + \frac{t^2}{2} \;\; (0 \le t \le 3)$

- $x' = -\sin t$
- $y' = 1 + \cos t$

$$\int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$
$$= \int_0^{\pi} \sqrt{\sin^2 t + 1 + 2\cos t + \cos^2 t}\, dt$$
$$= \int_0^{\pi} \sqrt{2(1+\cos t)}\, dt$$
$$= \int_0^{\pi} \sqrt{2^2 \cos^2 \frac{t}{2}}\, dt$$
$$= \int_0^{\pi} 2\cos\frac{t}{2}\, dt = \left[4 \sin \frac{t}{2}\right]_0^{\pi}$$
$$= 4$$

$$\cos 2 \cdot \frac{t}{2} = 2\cos^2 \frac{t}{2} - 1$$
$$\Rightarrow 1 + \cos t = 2\cos^2 \frac{t}{2}$$

- $x' = (2t+3)^{1/2}$
- $y' = 1 + t$

$$\int_0^{3} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$
$$= \int_0^{3} \sqrt{(2t+3) + (1+t)^2}\, dt$$
$$= \int_0^{3} \sqrt{t^2 + 4t + 4}\, dt$$
$$= \int_0^{3} \sqrt{(t+2)^2}\, dt$$
$$= \int_0^{3} (t+2)\, dt = \left[\frac{t^2}{2} + 2t\right]_0^{3}$$
$$= \frac{9}{2} + 6$$

Length of Parametric Equations

Find the lengths of the curves

- $x = 8\cos t + 8t\sin t\,, y = 8\sin t - 8t\cos t\ (0 \leq t \leq \pi/2)$

- $x' = -8\sin t + 8\sin t + 8t\cos t$
- $y' = 8\cos t - 8\cos t + 8t\sin t$

$$\int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\pi/2} \sqrt{8^2 t^2 \sin^2 t + 8^2 t^2 \cos^2 t}\, dt$$

$$= \int_0^{\pi/2} \sqrt{8^2 t^2}\, dt$$

$$= \int_0^{\pi/2} 8t\, dt$$

$$= [4t^2]_0^{\pi/2}$$

$$= \pi^2$$

Surface area of Parametric Equations

Find the areas of the surfaces generated by revolving the curves

- $x = \cos t, y = 2 + \sin t \ (0 \le t \le 2\pi)$; x −axis
- $x = \ln(\sec t + \tan t) - \sin t, y = \cos t \ (0 \le t \le \pi/3)$; x −axis

$$2\pi \int_0^{2\pi} y(t)\sqrt{(x')^2 + (y')^2}dt$$
$$= \int_0^{2\pi} (2 + \sin t)\sqrt{\sin^2 t + \cos^2 t}\, dt$$
$$= 2\pi \int_0^{2\pi} 2dt$$
$$= 8\pi^2$$

$$2\pi \int_0^{\pi/3} y(t)\sqrt{(x')^2 + (y')^2}dt$$
$$= 2\pi \int_0^{\pi/3} \cos t \sqrt{(\sec t - \cos t)^2 + \sin^2 t}\, dt$$
$$= 2\pi \int_0^{\pi/3} \cos t \sqrt{\sec^2 t - 2 + \cos^2 t + \sin^2 t}\, dt$$
$$= 2\pi \int_0^{\pi/3} \cos t \sqrt{\sec^2 t - 1} dt$$
$$= 2\pi \int_0^{\pi/3} \cos t \tan t\, dt$$
$$= 2\pi \int_0^{\pi/3} \sin t\, dt$$
$$= 2\pi [-\cos t]_0^{\pi/3}$$
$$= 2\pi \left(-\frac{1}{2} + 1\right) = \pi$$

Polar coordinates

Polar vs Cartesian Coordinates

Find the Cartesian coordinates of the following points

- $\left(\sqrt{2}, \frac{\pi}{4}\right)$
- $(1,0)$

$x = r\cos\theta$
$y = r\sin\theta$

- $x = \sqrt{2}\cos\frac{\pi}{4} = 1$
- $y = \sqrt{2}\sin\frac{\pi}{4} = 1$

$x = r\cos\theta$
$y = r\sin\theta$

- $x = 1\cos 0 = 1$
- $y = 1\sin 0 = 0$

Find the polar coordinates, $0 \le \theta \le 2\pi$ and $r \ge 0$ of the following points given in Cartesian coordinates.

- $(\sqrt{3}, -1)$
- $(-3,4)$

$r = \sqrt{x^2 + y^2}$
$\theta = \operatorname{atan}\frac{y}{x}$

- $r = \sqrt{3+1} = 2$
- $\theta = \operatorname{atan}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

$r = \sqrt{x^2 + y^2}$
$\theta = \operatorname{atan}\frac{y}{x}$

- $r = \sqrt{9+14} = 5$
- $\theta = \operatorname{atan}\left(-\frac{3}{4}\right) = -\operatorname{atan}\left(\frac{3}{4}\right)$

Polar coordinates

Polar graphs

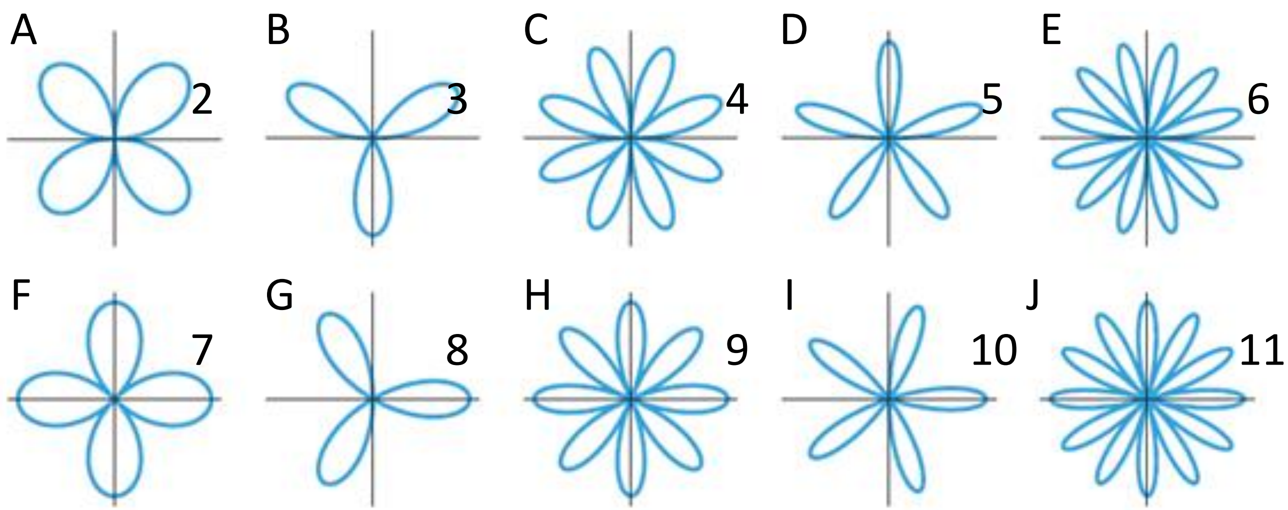
Graph the sets of points whose polar coordinates satisfy the equations and inequalities

- $r = 1$
- $1 \leq r \leq 2$
- $\theta = \pi/4$
- $\frac{\pi}{3} \leq \theta \leq \frac{\pi}{4}, 1 \leq r \leq 2$

Polar coordinates
Polar graphs

Match the polar equations with the graphs labeled

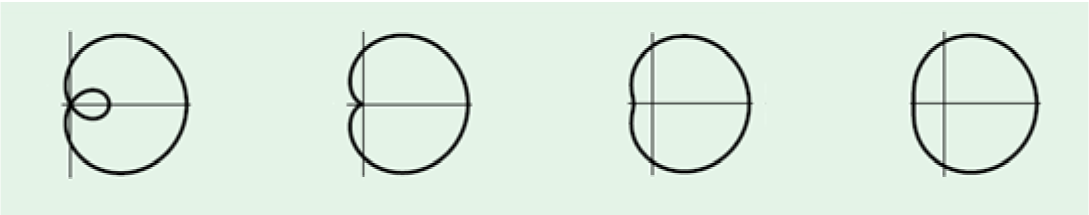
1. $r = \sin\theta$
2. $r = \sin 2\theta$
3. $r = \sin 3\theta$
4. $r = \sin 4\theta$
5. $r = \sin 5\theta$
6. $r = \sin 6\theta$
7. $r = \cos\theta$
8. $r = \cos 2\theta$
9. $r = \cos 3\theta$
10. $r = \cos 4\theta$
11. $r = \cos 5\theta$
12. $r = \cos 6\theta$



Match the polar equations with the graphs labeled

1. $r = 0.5 + \cos\theta$
2. $r = 1 + \cos\theta$
3. $r = 1.5 + \cos\theta$
4. $r = 2 + \cos\theta$

A B C D



1 2 3 4

Polar coordinates

Polar to Cartesian Equations

Find the Cartesian equations.

- $r = 4\tan\theta\sec\theta$
- $r\sin\theta = \ln r + \ln\cos\theta$
- $r = \cos\theta$

$r = 4\tan\theta\sec\theta$

$r = \frac{4\sin\theta}{\cos^2\theta}$

$r^2\cos^2\theta = 4r\sin\theta$

$x^2 = 4y$

- $r\sin\theta = \ln r + \ln\cos\theta$

$r\sin\theta = \ln r\cos\theta$

$y = \ln x$

- $r = \cos\theta$

$r^2 = r\cos\theta$

$x^2 + y^2 = x$