



Problem 1. Find the derivative (*i.e.* $\frac{df}{dx}$) for the following:

$$(1) f(x) = (4x^2 - 2x + 5)^{10}$$

$$f'(x) = 10(4x^2 - 2x + 5)^9 \cdot \left[\frac{d}{dx}(4x^2 - 2x + 5) \right]$$

$$= [10(4x^2 - 2x + 5)^9] \cdot (8x - 2)$$

$$(2) f(x) = 5x^3 e^x + 2^x$$

$$f'(x) = \left[(5x^3) \cdot (e^x) + \underbrace{(5 \cdot 3x^2)}_{15x^2} (e^x) \right] + 2^x \cdot \ln(2)$$

$$(3) f(x) = e^{5x^2}$$

$$f'(x) = e^{5x^2} \cdot (\cancel{\ln e}) \cdot \frac{d}{dx}(5x^2) = e^{5x^2} \cdot (10x)$$

#3

$$(4) f(x) = \log_2(x^2 e^{-x})$$

$$f'(x) = \frac{1}{(x^2 e^{-x}) \cdot \ln(2)} \cdot \frac{d}{dx} \left[\underbrace{x^2 e^{-x}}_{\text{base}} \right] .$$

$$f'(x) = \frac{1}{x^2 e^{-x} \cdot \ln(2)} \left[(2x) \cdot (e^{-x}) + (x^2) \left(e^{-x} \cdot \underbrace{\frac{d}{dx}(-x)}_{(-1)} \right) \right] .$$

$$(5) f(x) = \ln(\sqrt{x^2 - 5})$$

$$f'(x) = \frac{1}{\sqrt{x^2 - 5} (\ln e)} \cdot \underbrace{\frac{d}{dx}(\sqrt{x^2 - 5})}_{\frac{d}{dx}(x^2 - 5)^{1/2}} \quad \begin{matrix} \text{for } \ln(x), \text{ base } = e \\ \ln(e) = 1 \end{matrix}$$

$$\frac{d}{dx}(x^2 - 5)^{1/2} = \frac{1}{2}(x^2 - 5)^{-1/2} \left[\frac{d}{dx}(x^2 - 5) \right]$$

$$= \frac{1}{2}(x^2 - 5)^{-1/2} \cdot (2x)$$

$$f'(x) = \frac{1}{\sqrt{x^2 - 5}} \cdot \frac{1}{2} (x^2 - 5)^{-1/2} \cdot (2x)$$

$$= \underbrace{\frac{1}{2} (x^2 - 5)^{-1/2}}_{\frac{1}{x^2 - 5} = (x^2 - 5)^{-1}} \cdot (x^2 - 5)^{1/2} \cdot (2x)$$

$$= \frac{\cancel{x}^{x \rightarrow 0} \cancel{x \rightarrow 0} \cdot \ln(x)}{\frac{1}{x-5} = (x^2-5)^{-1}}$$

$$= \frac{x}{(x^2-5)}$$

$$\frac{1}{\sqrt{x^2-5}} = \frac{1}{(x^2-5)^{1/2}} = (x^2-5)^{-1/2}$$

$$\frac{1}{\sqrt{x}} \cdot \frac{1}{\sqrt{x}} = \frac{1}{x}$$

2

$$\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

quotient rule

$$(6) f(x) = \frac{1 + \ln(3x^2)}{1 + \ln(4x)}$$

$$\begin{aligned} f'(x) &= \frac{[1 + \ln(4x)] \frac{d}{dx}[1 + \ln(3x^2)] - [1 + \ln(3x^2)] \frac{d}{dx}[1 + \ln(4x)]}{[1 + \ln(4x)]^2} \\ &= \frac{[1 + \ln(4x)] \left[\frac{1}{3x^2} \cdot 2(2x) \right] - [1 + \ln(3x^2)] \left[\frac{1}{4x} \cdot 4 \right]}{[1 + \ln(4x)]^2} \end{aligned}$$

$$(7) f(x) = \underbrace{\log_5(3x^4 - 2x)}_{\text{underbrace}} e^{3x^2+1}$$

$$\begin{aligned} f'(x) &= \left[\frac{3(4x^3) - 2(1)}{\ln(5)(3x^4 - 2x)} \right] e^{3x^2+1} + \log_5(3x^4 - 2x) \left[e^{3x^2+1} \cdot (3 \cdot 2x) \right] \\ &= \frac{12x^3 - 2}{\ln 5 (3x^4 - 2x)} \cdot e^{3x^2+1} + \log_5(3x^4 - 2x) \cdot e^{3x^2+1} \cdot (6x) \end{aligned}$$

$$(8) f(x) = \sqrt[3]{e^{x^2} \ln(4x^2 + 2x)} = \underbrace{\left[e^{x^2} \cdot \ln(4x^2 + 2x) \right]^{1/3}}$$

$$\begin{aligned} f'(x) &= \frac{1}{3} \left[e^{x^2} \cdot \ln(4x^2 + 2x) \right]^{-2/3} \left[e^{x^2} \cdot (2x) \cdot \ln(4x^2 + 2x) + \right. \\ &\quad \left. e^{x^2} \cdot \frac{(4 \cdot 2x + 2 \cdot 1)}{(4x^2 + 2x)} \right]. \end{aligned}$$

$$\text{need: point } (x_0, y_0) \text{ + slope } m \xrightarrow{\text{Eqn}} y - y_0 = m(x - x_0)$$

Problem 2. Find the equation of the tangent line to the graph of $f(x) = e^{2x-3}$ at $x = 3/2$.

$$\begin{aligned} f(x) &= e^{2x-3} \\ f'(x) &= e^{2x-3} \cdot \frac{d}{dx}(2x-3) \\ m &= f'(x) \Big|_{x=3/2} = e^{2x-3} \cdot (2) \Big|_{x=3/2} = 2e^{2 \cdot \frac{3}{2} - 3} = 2e^{3-3} = 2e^0 = 2 \end{aligned}$$

$$\begin{aligned} y &= f\left(\frac{3}{2}\right) = e^{2 \cdot \frac{3}{2} - 3} \\ &= e^{3-3} \\ &= e^0 \\ &= 1 \end{aligned}$$

$$\text{point } (x_0, y_0) = \left(\frac{3}{2}, 1\right)$$

$$m = 2$$

$$\begin{aligned} y - 1 &= 2 \overbrace{(x - \frac{3}{2})}^+ = 2x - 2 \cdot \frac{3}{2} = 2x - 3 \\ y &= 2x - 3 \Rightarrow y = 2x - 2 \end{aligned}$$

Problem 3. if $f(x) = \ln(x^3 + 2)$ find $f'(e^{1/3})$.

$$\begin{aligned} f(x) &= \ln(x^3 + 2) \\ f'(x) &= \frac{1}{(x^3 + 2)} \cdot \frac{d}{dx}(x^3 + 2) = \frac{3x^2}{(x^3 + 2)} \end{aligned}$$

$$f'(e^{1/3}) = \frac{3 \cdot (e^{1/3})^2}{(e^{1/3})^3 + 2} = \frac{3e^{2/3}}{e + 2}$$

$$e^{2/3} = e^1 = e$$

when you cannot write
 $y = f(x)$ → cannot isolate y on one side only

4

↓ f

Problem 4. Use implicit differentiation to find $\frac{dy}{dx}$ for the following:

$$(1) x^2 + xy + y^2 = 7$$

$$\begin{aligned} & \overbrace{x^2 + xy + y^2}^{f(x,y)} = 7 \\ & \frac{d}{dx} [x^2 + xy + y^2] = \frac{d}{dx} [7] \end{aligned}$$

Rules for implicit

$$① \frac{d}{dx}(x) = 1$$

$$② \frac{d}{dx}(y) \rightarrow \text{solve for it}$$

$$① \frac{d}{dx}(x^2) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = 0$$

$$2x + \underbrace{\left[1 \cdot y + x \cdot \frac{dy}{dx} \right]}_{\frac{dy}{dx}} + 2y \cdot \frac{dy}{dx} = 0$$

$$② x \frac{dy}{dx} + 2y \frac{dy}{dx} = -2x - y$$

$$\frac{dy}{dx}(x + 2y) = -2x - y$$

$$③ \frac{dy}{dx} = \frac{-2x - y}{x + 2y} \quad \text{Ans.}$$

$$(2) \sqrt{x} + \sqrt{y} = 16$$

$$\frac{d}{dx}(\sqrt{x}) + \frac{d}{dx}(\sqrt{y}) = \frac{d}{dx}(16)$$

$$① \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}} \cdot \frac{dy}{dx} = 0$$

$$② \frac{1}{2}y^{-\frac{1}{2}} \frac{dy}{dx} = -\frac{1}{2}x^{-\frac{1}{2}}$$

$$③ \frac{dy}{dx} = \frac{\frac{1}{2}x^{-\frac{1}{2}}}{\frac{1}{2}y^{-\frac{1}{2}}} = -\frac{x^{-\frac{1}{2}}}{y^{-\frac{1}{2}}} = -\frac{\sqrt{x}}{\sqrt{y}}$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}x^{\frac{1}{2}}$$

$$= \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}}$$

$$= \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}(e^{xy}) = e^{xy} \cdot \frac{d}{dx}(xy)$$

$$(3) x^2 + ye^{xy} = 1$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(ye^{xy}) = \frac{d}{dx}(1)$$

$$= 2x + \left[\frac{dy}{dx} \cdot e^{xy} + y \cdot e^{xy} \left(\frac{d}{dx}(xy) \right) \right] = 0$$

$$(ye^{xy})(1 \cdot y + x \frac{dy}{dx})$$

$$= 2x + e^{xy} \frac{dy}{dx} + y \cdot e^{xy} \left(y + x \frac{dy}{dx} \right) = 0 .$$

$$= 2x + \cancel{e^{xy} \frac{dy}{dx}} + \cancel{y^2 \cdot e^{xy}} + \cancel{x \cdot e^{xy} \cdot \frac{dy}{dx}} = 0 \quad \left. \begin{array}{l} \frac{dy}{dx} = \frac{-2x - y^2 \cdot e^{xy}}{e^{xy} + xy \cdot e^{xy}} \\ \end{array} \right\}$$

$$\frac{dy}{dx} (e^{xy} + xy \cdot e^{xy}) = -2x - y^2 \cdot e^{xy}$$

$$(4) \log_{10}(5y^4) - e^{x^2y} = 10$$

$$\underbrace{\frac{d}{dx} [\log_{10}(5y^4)]}_{\frac{d}{dx}(5y^4)} - \underbrace{\frac{d}{dx}[e^{x^2y}]}_{e^{x^2y} \cdot \frac{d}{dx}(x^2y)} = \underbrace{\frac{d}{dx}(10)}_0$$

$$\frac{\frac{d}{dx}(5y^4)}{5y^4 \cdot \ln(10)}$$

$$e^{x^2y} \left[2x \cdot y + x^2 \cdot \frac{dy}{dx} \right]$$

$$\frac{1}{5y^4 \cdot \ln(10)} \cdot 5 \cdot 4y^3 \cdot \frac{dy}{dx} - \left[\frac{4}{y \ln(10)} \cdot \frac{dy}{dx} \right] - \left[2xy \cdot e^{x^2y} + x^2 \cdot e^{x^2y} \cdot \frac{dy}{dx} \right] = 0$$

$$\textcircled{2} \quad \frac{dy}{dx} \left[\frac{4}{y \ln(10)} - x^2 e^{x^2y} \right] = +2xy e^{x^2y}$$

$$\textcircled{3} \quad \left| \frac{dy}{dx} = \frac{2xy e^{x^2y}}{\frac{4}{y \ln(10)} - x^2 e^{x^2y}} \right. \text{ Ans.}$$

$m + \uparrow (x_0, y_0)$

Problem 5. Differentiate implicitly and find the equation of the tangent line at the given point.

$$(1) x^2 + y^2 = 4 \text{ at the point } \underbrace{(1, -\sqrt{3})}_{x_0=1, y_0=-\sqrt{3}}$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(4)$$

$$2x + 2y \cdot \frac{dy}{dx} = 0 \rightarrow 2y \frac{dy}{dx} = -2x$$

$$m = \frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y} \Big|_{\begin{array}{l} x=1 \\ y=-\sqrt{3} \end{array}} = -\frac{1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$y + \sqrt{3} = \frac{1}{\sqrt{3}}(x - 1)$$

$$y = \frac{x}{\sqrt{3}} - \frac{1}{\sqrt{3}} - \sqrt{3}$$

$$(2) \ln(xy) = y^2 - 1 \text{ when } x = 1, y = 1$$

(1, 1)

$$\begin{aligned} y - 1 &= 1(x - 1) \\ y - 1 &= x - 1 \\ \boxed{y = x} \end{aligned}$$

$$\underbrace{\frac{d}{dx} \ln(xy)}_{\frac{1}{xy} \cdot \frac{d}{dx}(xy)} = \underbrace{\frac{d}{dx}(y^2)}_{2y \cdot \frac{dy}{dx}} - \frac{d}{dx}(1)$$

$$= \frac{1}{xy} \cdot \frac{d}{dx}(xy) = 2y \cdot \frac{dy}{dx} - 0$$

$$= \frac{1}{xy} \left[1 \cdot y + x \frac{dy}{dx} \right]$$

$$= \frac{1}{xy} + \frac{x}{xy} \frac{dy}{dx}$$

$$\frac{1}{x} + \frac{1}{y} \frac{dy}{dx} = 2y \frac{dy}{dx}$$

$$\frac{1}{x} = 2y \frac{dy}{dx} - \frac{1}{y} \frac{dy}{dx}$$

$$\frac{1}{x} = \frac{dy}{dx} (2y - \frac{1}{y})$$

$$m = \frac{dy}{dx} = \frac{\frac{1}{x}}{2y - \frac{1}{y}} \Big|_{\begin{array}{l} x=1 \\ y=1 \end{array}}$$

$$= \frac{\frac{1}{1}}{2(1) - \frac{1}{1}} = \frac{1}{2-1} = 1$$

$$\begin{aligned} x(x+1) &= x^2+x \\ x(x+1)^2 &\neq (x^2+x)^2 \\ x(\cancel{x^2+2x+1}) &= \cancel{x^3+2x^2+x} \end{aligned}$$

Problem 6. The price demand function, $p(x) = -(-0.01x - 2)^4 + 303$, is for a brand of towels. $p(x)$ is the price in dollars, per towel, when there is a demand for x towels.

(1) Find the marginal revenue function.

$$\begin{aligned} R'(x) &= x p(x) = (x) [-(-0.01x - 2)^4 + 303] \\ &= -x (-0.01x - 2)^4 + 303x \end{aligned}$$

$$R'(x) = [(-1)(-0.01x - 2)^4 + (-x) \cdot 4(-0.01x - 2)^3 \cdot (-0.01)] + 303$$

$$R'(x) = -(-0.01x - 2)^4 + 0.04x(-0.01x - 2)^3 + 303 \quad \$/\text{towel}$$

(2) Find the marginal revenue when 150 towels are sold. Interpret your answer.

$$R'(x=150) \approx -104.31$$

when 150 towels are sold, revenue is decreasing at a rate of \$104.31 per towel.

Problem 7. A bank account has an initial balance of \$400. The account earns interest at an annual rate of 3.24% per year compounded continuously. How fast is the account balance growing after 7 years?

$$t$$

$$r = 0.0324$$

$$A(t) = Pe^{rt}$$

$$A'(t) = ?$$

$$A'(t) = \frac{d}{dt} Pe^{rt} = P \cdot \frac{d}{dt} e^{rt} = P \cdot e^{rt} \cdot \frac{d}{dt}(rt)$$

$$A'(t) = P \cdot r \cdot e^{rt}$$

$$A'(7) = (400)(0.0324) e^{(0.0324)(7)}$$

$$\approx 16.26 \text{ \$/year.}$$

for rate
 $\frac{d}{dt}(t) = 1$
 $\frac{d}{dt}(0) = 0$
 $\frac{d}{dt}(r) = 0$ $\Rightarrow r$ are constants

Problem 8. Use the table below to answer the following questions

x	-6	0	5	8	64
$f(x)$	30	-6	19	58	4090
$f'(x)$	-12	0	10	16	128
$g(x)$	24	0	35	80	4224
$g'(x)$	-10	2	12	18	130

(1) Find $h'(5)$ if $h(x) = x^2 - 3(g(x))^4$

$$h'(x) = 2x - 3 \cdot 4(g(x))^3 \cdot g'(x)$$

$$\begin{aligned} h'(5) &= 2(5) - 12(g(5))^3 \cdot g'(5) \\ &= 10 - 12(35)^3 \cdot (12) \end{aligned}$$

$$= -6,173,990$$

(2) Find $h'(8)$ if $h(x) = \underbrace{5f(x)}_{\text{underbrace}} \underbrace{g(x^2)}_{\text{underbrace}}$

$$h'(x) = 5f'(x) \cdot g(x^2) + 5f(x)g'(x^2) \cdot 2x$$

$$\begin{aligned} h'(8) &= 5f'(8)g(64) + 10f(8)g'(64) \cdot 8 \\ &= 5(16)(4224) + 80(58)(130) \\ &= 941,120 \end{aligned}$$

(3) Find $h'(x)$ if $h(x) = \overbrace{f(g(f(x) + 3))}^{\text{underbrace}}$

$$h'(x) = f'(g(f(x) + 3)) \cdot g'(f(x) + 3) \cdot f'(x)$$