Note ♯1: Sections 12.1-12.5

Problem 1. a) Find the center and radius of the sphere $x^2 + y^2 + z^2 - x + 4y - 10z - 1 = 0$.

"completion of squares"
$x^2 + 2ax = (x+a)^2 - a^2$

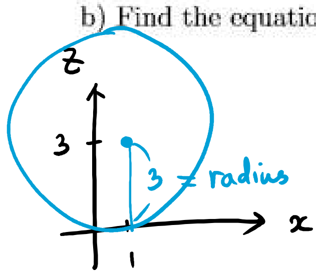
$$\Leftrightarrow (x^2 - x) + (y^2 + 4y) + (z^2 - 10z) - 1 = 0$$

$$\Leftrightarrow \left(x - \tfrac{1}{2}\right)^2 - \tfrac{1}{4} + (y+2)^2 - 4 + (z-5)^2 - 25 - 1 = 0$$

$$\Leftrightarrow \boxed{\left(x - \tfrac{1}{2}\right)^2 + (y+2)^2 + (z-5)^2} = \tfrac{1}{4} + 4 + 25 + 1 = \boxed{\tfrac{121}{4}}$$

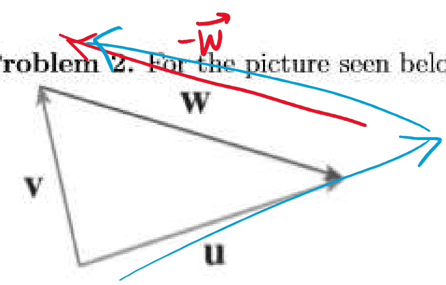
$\boxed{\text{center}: \left(\tfrac{1}{2}, -2, 5\right)}$ $\boxed{\text{radius} = \sqrt{\tfrac{121}{4}} = \tfrac{11}{2}}$

b) Find the equation of the sphere with center $(1, 4, 3)$ that touches the xy plane.



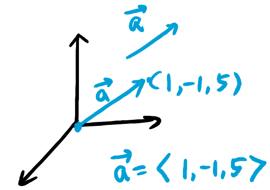
$$(x-1)^2 + (y-4)^2 + (z-3)^2 = 3^2 = 9$$

Problem 2. For the picture seen below, write $\mathbf{v}$ in terms of $\mathbf{u}$ and $\mathbf{w}$.



$$\vec{v} = \vec{u} - \vec{w}$$

length 1 : $\vec{a}/|\vec{a}|$



Problem 3. Given $\mathbf{a} = \langle 1, -1, 5\rangle$ and $\mathbf{b} = \langle -3, 2, 1\rangle$,

a) find a unit vector in the direction of $\mathbf{a} + 2\mathbf{b}$.

$$\vec{a} + 2\vec{b} = \langle 1, -1, 5\rangle + \underbrace{2\langle -3, 2, 1\rangle}_{\langle -6, 4, 2\rangle} = \langle -5, 3, 7\rangle$$

$$|\langle -5, 3, 7\rangle| = \sqrt{5^2 + 3^2 + 7^2} = \sqrt{25 + 9 + 49} = \sqrt{83}$$

$$\boxed{\frac{\langle -5, 3, 7\rangle}{\sqrt{83}} = \left\langle -\frac{5}{\sqrt{83}}, \frac{3}{\sqrt{83}}, \frac{7}{\sqrt{83}}\right\rangle}$$

b) find the vector that has the same direction as $\mathbf{a} + 2\mathbf{b}$ but has length 4.

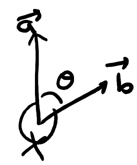
$$= 4(\vec{a} + 2\vec{b}) = \boxed{\frac{4}{\sqrt{83}}\langle -5, 3, 7\rangle}$$

Problem 4. Compute $\mathbf{a} \cdot \mathbf{b}$ if

$$\langle a_1, a_2, a_3\rangle \cdot \langle b_1, b_2, b_3\rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$$

a) $\mathbf{a} = \langle 4, 5, -1\rangle$ and $\mathbf{b} = \langle 2, 1, 3\rangle$.

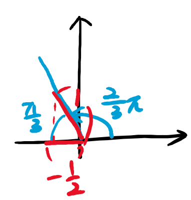
$$\vec{a} \cdot \vec{b} = 4 \cdot 2 + 5 \cdot 1 + (-1)(3) = 8 + 5 - 3 = 10$$



b) $|\mathbf{a}| = 2$, $|\mathbf{b}| = 5$ and $\theta = 120°$.

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta,$$

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos(\theta) = 2 \cdot 5 \cdot \cos(120°) = 2 \cdot 5 \cdot \left(-\tfrac{1}{2}\right) = -5$$



$\theta = \frac{\pi}{2}$

c) $|\mathbf{a}| = 6$, $|\mathbf{b}| = 4$ and $\mathbf{a}$ is perpendicular to $\mathbf{b}$.

$$\vec{a} \cdot \vec{b} = 6 \cdot 4 \cdot \cos\left(\tfrac{\pi}{2}\right) = 6 \cdot 4 \cdot 0 = 0$$

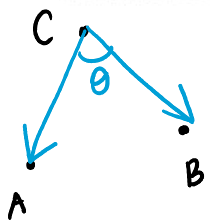
$\boxed{\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a}, \vec{b} \text{ perpendicular (orthogonal)}}$

$\theta = 0$

d) $|\mathbf{a}| = 6$, $|\mathbf{b}| = 4$ and $\mathbf{a}$ is parallel to $\mathbf{b}$.

$$\vec{a} \cdot \vec{b} = 6 \cdot 4 \cdot \cos(0) = 6 \cdot 4 \cdot 1 = 24$$

Problem 5. The points $A(0,-1,6)$, $B(2,1,-3)$ and $C(5,4,2)$ form a triangle. Find $\angle C$.



$\vec{CA} = A - C = \langle 0,-1,6\rangle - \langle 5,4,2\rangle = \langle -5,-5,4\rangle$

$|\vec{CA}| = \sqrt{5^2+5^2+4^2} = \sqrt{66}$

$\vec{CB} = B - C = \langle -3,-3,-5\rangle$

$|\vec{CB}| = \sqrt{3^2+3^2+5^2} = \sqrt{43}$

$\vec{CA}\cdot\vec{CB} = |\vec{CA}||\vec{CB}|\cos\theta$

$\cos\theta = \dfrac{10}{\sqrt{66}\sqrt{43}}$

$\vec{CA}\cdot\vec{CB} = (-5)(-3)+(-5)(-3)+(4)(-5) = 10$

$\boxed{\theta = \cos^{-1}\left(\dfrac{10}{\sqrt{66}\sqrt{43}}\right)}$

Can it be calculated?

Problem 6. Let **a**, **b**, and **c** be three dimensional vectors. Which of the following expressions are meaningful? Which are meaningless?

a) $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})$ = real number exists ⇒ meaningful

3d · 3d

b) $\mathbf{a}\cdot\mathbf{b}+\mathbf{c}$ ⇒ meaningless

3d · 3d   3d ✗

= 1d +

$\vec{a}\cdot\vec{b} \overset{?}{=} 0$ $\vec{a} \overset{?}{=} c\vec{b}$

Problem 7. Determine whether the given vectors are orthogonal, parallel, or neither.

a) $\mathbf{a} = \langle 3,-1,2\rangle$, $\mathbf{b} = \langle 6,-2,4\rangle$

×2 ×2 ×2

$\vec{b} = 2\vec{a}$ Parallel

b) $\mathbf{a} = \langle 1,2,-1\rangle$, $\mathbf{b} = \langle 2,3,-1\rangle$ not parallel

×2 ×3/2

$\vec{a}\cdot\vec{b} = 1\cdot 2 + 2\cdot 3 + (-1)(-1) = 2+6+1 \neq 0$ not orthogonal

neither

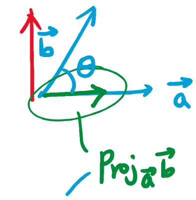
c) $\mathbf{a} = 2\mathbf{i}-\mathbf{j}+2\mathbf{k}$, $\mathbf{b} = -2\mathbf{i}+2\mathbf{j}+3\mathbf{k} = \langle -2,2,3\rangle$

$\vec{a}\cdot\vec{b} = 2(-2)+(-1)(2)+(2)(3) = -4-2+6 = 0$ ⇒ orthogonal

Problem 8. Find the scalar and vector projections of $\langle 2,4,6\rangle$ onto $\langle 1,3,5\rangle$.

$\text{comp}_{\vec{a}}\vec{b} = \boxed{\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}$ $\qquad \vec{b} = \langle 2,4,6\rangle \qquad \vec{a} = \langle 1,3,5\rangle$

$\boxed{\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}}$



directed length is $\text{comp}_{\vec{a}}\vec{b}$

$$\vec{a}\cdot\vec{b} = 1\cdot 2 + 3\cdot 4 + 5\cdot 6 = 2+12+30$$
$$= 44$$

$$|\vec{a}| = \sqrt{1^2+3^2+5^2} = \sqrt{35}$$

$$\boxed{\text{com}_{\vec{a}}\vec{b} = \frac{44}{\sqrt{35}}} \qquad \boxed{\text{Proj}_{\vec{a}}\vec{b} = \frac{44}{35}\langle 1,3,5\rangle}$$

Problem 9. Find the cross product of $\langle 1,1,3\rangle$ and $\langle -2,-1,-5\rangle$.

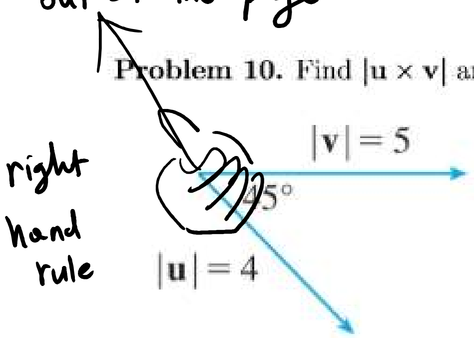
$$\langle 1,1,3\rangle \times \langle -2,-1,-5\rangle = \begin{vmatrix} +\hat{\imath} & -\hat{\jmath} & +\hat{k} \\ 1 & 1 & 3 \\ -2 & -1 & -5 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ -1 & -5 \end{vmatrix}\hat{\imath} - \begin{vmatrix} 1 & 3 \\ -2 & -5 \end{vmatrix}\hat{\jmath} + \begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix}\hat{k}$$

$$= \big((1)(-5)-(3)(-1)\big)\hat{\imath} - \big((1)(-5)-(3)(-2)\big)\hat{\jmath} + \big((1)(-1)-(1)(-2)\big)\hat{k}$$

$$= \boxed{-2\hat{\imath} - \hat{\jmath} + \hat{k}}$$

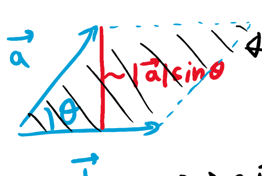
out of the page

Problem 10. Find $|\mathbf{u}\times\mathbf{v}|$ and determine if $\mathbf{u}\times\mathbf{v}$ points in or out of the page.



$$|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$$
$$= 4\cdot 5\cdot\sin(45°)$$
$$= 4\cdot 5\cdot\frac{\sqrt{2}}{2}$$
$$= \boxed{10\sqrt{2}}$$

Problem 11. Find the area of the parallelogram determined by $\mathbf{a} = \langle 3,0,2\rangle$ and $\mathbf{b} = \langle 1,-4,5\rangle$.



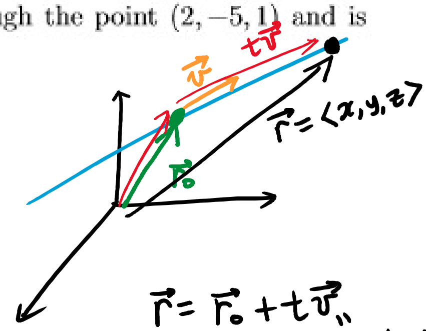
$$|\vec{a}\times\vec{b}| = \cdots = \sqrt{8^2+13^2+12^2}$$

$$|\vec{a}|\sin\theta\,|\vec{b}| = |\vec{a}||\vec{b}|\sin\theta = |\vec{a}\times\vec{b}|$$

Problem 12. Find a vector equation of the line that passes through the point $(2, -5, 1)$ and is parallel to the vector $\langle 8, 10, -7\rangle$.

$$\vec{r} = \underbrace{\langle 2,-5,1\rangle}_{\text{pt on the line}} + t\,\underbrace{\langle 8,10,-7\rangle}_{\text{vector parallel to the line}}$$

$\vec{r} = \langle x,y,z\rangle$



$\vec{r} = \vec{r_0} + t\vec{v}$, where $\vec{r} = \langle x,y,z\rangle$, $\vec{r_0} = \langle x_0,y_0,z_0\rangle$, $\vec{v} = \langle a,b,c\rangle$

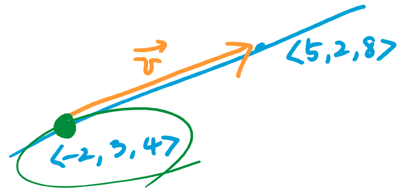
Problem 13. Find parametric equations and a symmetric equation for the line passing through the points $(-2, 3, 4)$ and $(5, 2, 8)$.

$$\langle x, y, z\rangle = \langle -2, 3, 4\rangle + t\langle 7, -1, 4\rangle$$
$$= \langle -2+7t,\ 3-t,\ 4+4t\rangle$$

$t = \frac{x+2}{7}$

$$\boxed{\begin{cases} x = -2+7t \\ y = 3-t \\ z = 4+4t \end{cases}\quad \text{Parametric Eqns}}$$

$$\boxed{t = \frac{x+2}{7} = \frac{y-3}{-1} = \frac{z-4}{4}\quad \text{Symmetric Eqn}}$$



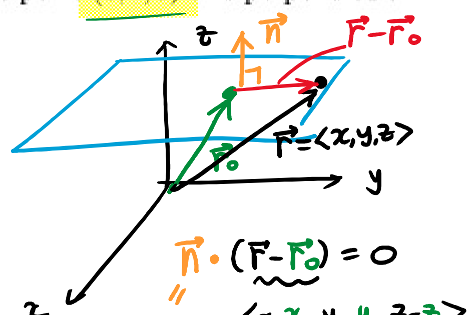
$\vec{v} = \langle 5,2,8\rangle - \langle -2,3,4\rangle$
$= \langle 7,-1,4\rangle$

Problem 14. Find an equation of the plane passing through the point $(3, 4, 5)$ and perpendicular to $\langle -1, 2, 5\rangle$.

$$\boxed{-1(x-3) + 2(y-4) + 5(z-5) = 0}$$

$\Leftrightarrow\ -x+3+2y-8+5z-25=0$

$$\Leftrightarrow\ \boxed{-x+2y+5z-30=0}\ ✓$$

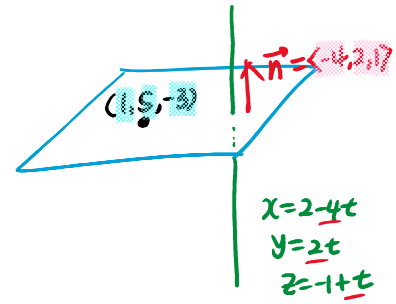


$\vec{n}\cdot(\vec{r}-\vec{r_0}) = 0$, where $\vec{n} = \langle a,b,c\rangle$, $\vec{r}-\vec{r_0} = \langle x-x_0,\ y-y_0,\ z-z_0\rangle$

$$\boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0}$$

Problem 15. Find an equation of the plane passing through the point $(1, 5, -3)$ and perpendicular to the line $x = 2 - 4t, y = 2t, z = -1 + t$.

$$\boxed{-4(x-1)+2(y-5)+(z+3)=0}$$

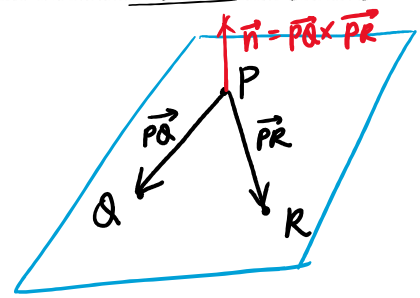


Problem 16. Find the equation of the plane that passes through the points $P(1, 0, 1)$, $Q(2, 3, 4)$ and $R(2, 1, 1)$.

$\vec{PQ} = Q - P = \langle 2,3,4\rangle - \langle 1,0,1\rangle = \langle 1,3,3\rangle$

$\vec{PR} = R - P = \langle 2,1,1\rangle - \langle 1,0,1\rangle = \langle 1,1,0\rangle$

$$\vec{n} = \vec{PQ}\times\vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 3 \\ 1 & 1 & 0 \end{vmatrix} = -3\hat{i}+3\hat{j}-2\hat{k}$$



$$-3(x-1)+3(y-0)-2(z-1)=0$$

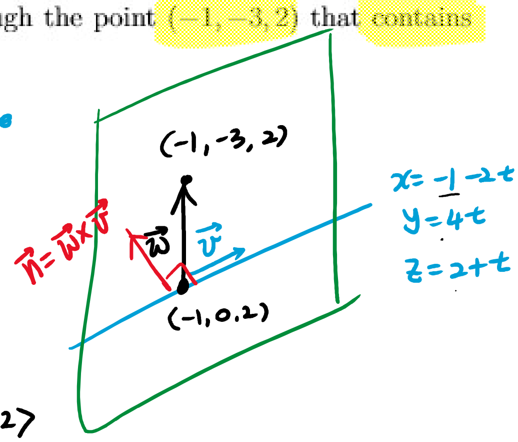
Problem 17. Find an equation of the plane passing through the point $(-1, -3, 2)$ that contains the line $x = -1 - 2t$, $y = 4t$, $z = 2 + t$.

$\vec{v} = \langle -2,4,1\rangle$ - direction vector of the line

$\vec{w} = \langle -1,-3,2\rangle - \underbrace{\langle -1,0,2\rangle}_{t=0 \text{ on the line}} = \langle 0,-3,0\rangle$

$$\vec{n} = \vec{w}\times\vec{v} = \begin{vmatrix} +\hat{i} & -\hat{j} & +\hat{k} \\ 0 & -3 & 0 \\ -2 & 4 & 1 \end{vmatrix} = -3\hat{i}-6\hat{k}$$

$= -3\langle 1,0,2\rangle$



$$\boxed{-3(x+1)-6(z-2)=0}$$

Problem 18. Consider the lines $\mathbf{r}_1(t) = \langle 2+t, 2t, 5+t\rangle$ and $\mathbf{r}_2(s) = \langle s, -4+4s, 3+s\rangle$.

a) Find the point of intersection of the lines

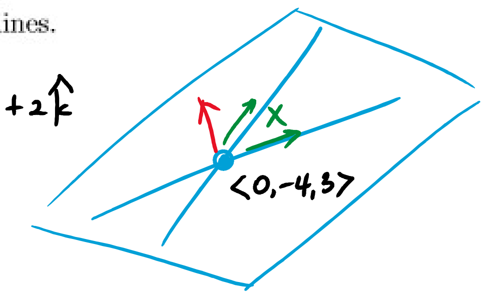
$$\begin{cases} x: & 2+t=s \iff t-s=-2 \cdots ① \\ y: & 2t=-4+4s \iff t-2s=-2 \cdots ② \\ z: & 5+t=3+s \iff t-s=-2 \end{cases}$$

$①-②$ $s=0 \Rightarrow t=-2$

$\vec{r}_2(0) = \boxed{\langle 0,-4,3\rangle}$

b) Find an equation of the plane that contains these lines.

$$\langle 1,2,1\rangle \times \langle 1,4,1\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ 1 & 4 & 1 \end{vmatrix} = -2\hat{i} + 2\hat{k}$$



$$-2(x-0) + 0(y+4) + 2(z-3) = 0$$

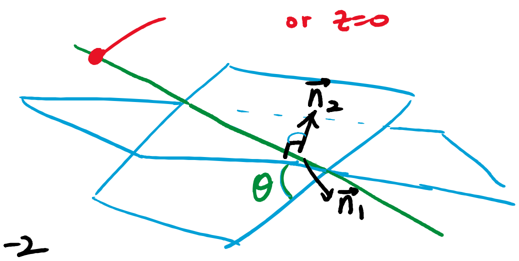
$$\iff \boxed{-2x + 2(z-3) = 0}$$

Problem 19. Consider the planes $z = x + y$ and $2x - 5y - z = 1$. Set $x=0$, or $y=0$ or $z=0$

a) Find the angle between the planes.

$z = x+y \iff x+y-z=0 \Rightarrow \vec{n}_1 = \langle 1,1,-1\rangle$

$2x-5y-z=1 \Rightarrow \vec{n}_2 = \langle 2,-5,-1\rangle$



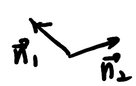
$\vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1||\vec{n}_2| \cos\theta$

$\vec{n}_1 \cdot \vec{n}_2 = (1)(2) + (1)(-5) + (-1)(-1) = 2-5+1 = -2$

$|\vec{n}_1| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$ $\quad |\vec{n}_2| = \sqrt{2^2+5^2+1^2} = \sqrt{30}$

$\cos\theta = \dfrac{+2}{\sqrt{3}\sqrt{30}}$ $\quad \boxed{\theta = \cos^{-1}\left(\dfrac{+2}{\sqrt{3}\sqrt{30}}\right)}$

$\vec{n}_1 \cdot \vec{n}_2 > 0$



$\vec{n}_1 \cdot \vec{n}_2 < 0$

b) Find the line of intersection of the planes.

vector parallel to the line $-$ $\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 2 & -5 & -1 \end{vmatrix} = -6\hat{i} - \hat{j} - 7\hat{k}$

a point on the line: $x=0$: $\begin{cases} z=y \\ -5y-z=1 \end{cases}$ $\quad -6y=1 \iff y = -\frac{1}{6} = z$ $\quad (0, -\frac{1}{6}, -\frac{1}{6})$

$$\boxed{\langle x,y,z\rangle = \langle 0, -\tfrac{1}{6}, -\tfrac{1}{6}\rangle + t\langle -6,-1,-7\rangle}$$