Math 308: Week-in-Review 3

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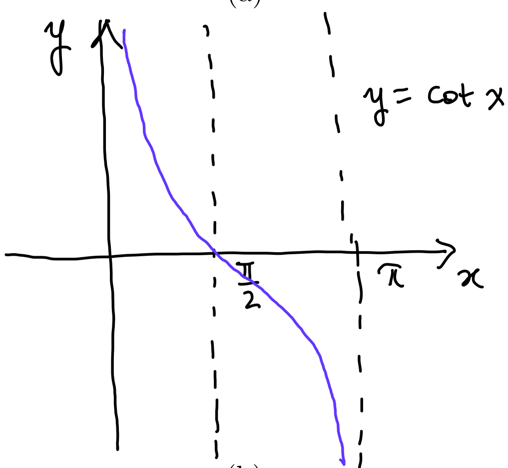
1. Determine (without solving the problem) an interval in which the solution of the following initial value problem is certain to exist.

* $y' + p(x)y = g(x)$ * standard first order linear
* $p(x), g(x)$ continuous on an interval I $\Rightarrow$ there exists a unique solution on I satisfying the initial condition

(a)

$$y' + (\cot x)y = x, \quad \underbrace{y(\pi/2) = 9}_{IC}$$



* $\cot x$ is discontinuous at all points $x = \pm n\pi$ for integer n
* the solution with IC $y(\pi/2) = 9$ exists in the interval $I = (0, \pi)$ where $\cot x$ is continuous

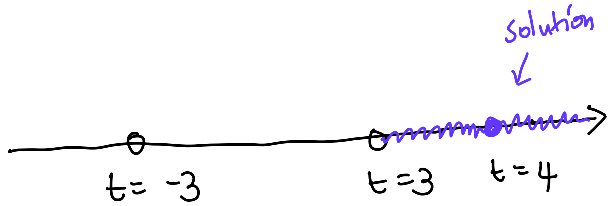
(b)

$$(t^2 - 9)y' + ty = t^4, \quad y(4) = 2$$

* rewrite in standard form $y' + p(t)y = g(t)$

$$y' + \frac{t}{t^2-9}y = \frac{t^4}{t^2-9}$$

* $p(t) = \frac{t}{t^2-9} = \frac{t}{(t-3)(t+3)}$ is discontinuous at $t = 3, t = -3$
* find points where $p(t), g(t)$ are discontinuous



* interval containing initial condition

$I = (3, \infty)$

(c)

$$y' = y\sqrt{t-3}, \quad y(1) = 2$$

* in standard form $y' + p(t)y = g(t)$
* $y' - \sqrt{t-3}\,y = 0$
* $p(t) = -\sqrt{t-3}$ is defined and continuous on $I = [3, \infty)$
* but $t = 1$ is NOT in I
* initial value problem has NO SOLUTION

2. State where in the ty-plane the hypothesis of the Existence and Uniqueness Theorem are satisfied for the following differential equations

* Existence & Uniqueness Thm

$y' = f(x,y)$ has a unique solution on some interval where

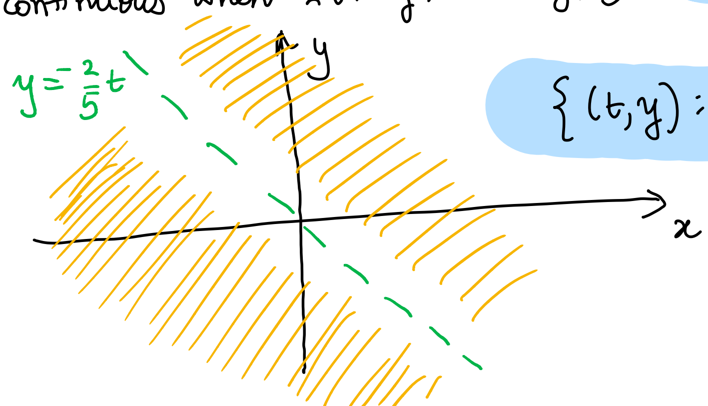
* $f(x,y)$ is continuous
* $\frac{\partial f}{\partial y}(x,y)$ is continuous

$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$ quotient rule

(a) $(2t + 5y)y' = t - y$

$y' = \frac{t-y}{2t+5y}$, $f(t,y) = \frac{t-y}{2t+5y}$

* f is continuous when $2t+5y \neq 0$, i.e. $y \neq -\frac{2}{5}t$
* $\frac{\partial f}{\partial y} = \frac{-(2t+5y) - 5(t-y)}{(2t+5y)^2} = \frac{-7t}{(2t+5y)^2}$
* $\frac{\partial f}{\partial y}$ is continuous when $2t+5y \neq 0$ i.e. $y \neq -\frac{2}{5}t$



$\{(t,y) : y \neq -\frac{2}{5}t\}$ ← EUT satisfied in this region

all points minus the line $y = -\frac{2}{5}t$

(b) $y' = \frac{\ln(ty)}{y - t^2}$

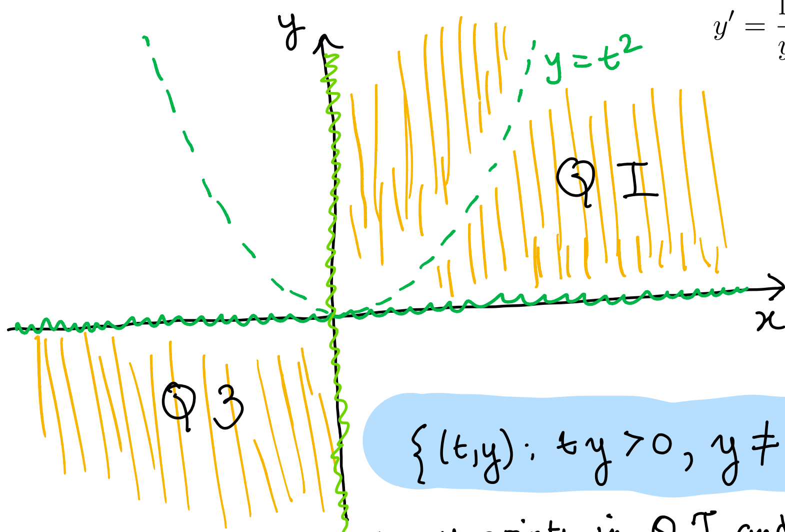
$f(t,y) = \frac{\ln(ty)}{y - t^2}$

* f defined when $ty > 0$ and $y \neq t^2$
* $\frac{\partial f}{\partial y} = \frac{t \cdot \frac{1}{ty}(y - t^2) - \ln(ty)}{(y-t^2)}$

$= \frac{y - t^2 - y\ln(ty)}{y(y-t^2)}$

* $\frac{\partial f}{\partial y}$ defined when $ty > 0$, $y \neq t^2$ and $y \neq 0$

not x-axis



$\{(t,y) : ty > 0, y \neq t^2\}$

* all points in Q I and Q3 except the axes and the curve $y = t^2$

3. Solve the following initial value problems and determine how the interval in which the solution exists depends on y_0.

   (a) $y' = y^2, \quad \underbrace{y(0) = y_0}_{\text{IC}}$ nonlinear, separable

$$\int \frac{1}{y^2}\,dy = \int dx$$

$$-\frac{1}{y} = x + C$$

plug in IC

$$C = -1/y_0$$

$$\frac{-1}{y} = x - \frac{1}{y_0}$$

$$y = \frac{-1}{x - \frac{1}{y_0}} = \frac{y_0}{1 - xy_0}$$

$$\boxed{y = \frac{y_0}{1 - xy_0}}$$

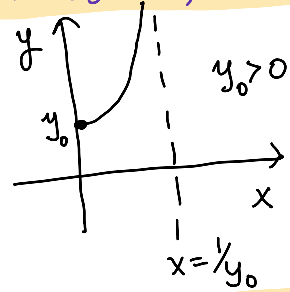
Equilibrium soln

$y^2 = 0 \Rightarrow y(t) = 0$

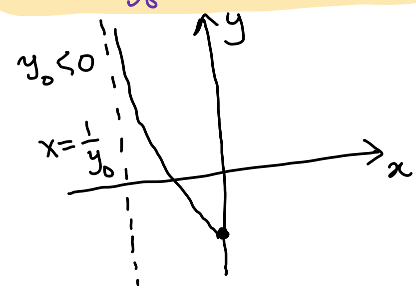
and $y(0) = y_0 = 0$

for all $x \in \mathbb{R}$

$*$ if $y_0 > 0$, then $x < \frac{1}{y_0}$



$*$ if $y_0 < 0$, then $x > \frac{1}{y_0}$



   (b) $y' = -\frac{4t}{y}, \quad \underbrace{y(0) = y_0}_{\text{IC}}$, nonlinear separable

$y \neq 0$

$y_0 \neq 0$ since $y \neq 0$

$$\int y\,dy = \int -4t\,dt$$

$$\frac{y^2}{2} = -4\frac{t^2}{2} + C$$

$$y^2 + 4t^2 = C$$

IC $\quad y_0^2 = C$

$$\Rightarrow y^2 + 4t^2 = y_0^2$$

$$y = \pm\sqrt{y_0^2 - 4t^2}$$

y is defined when

$$y_0^2 - 4t^2 > 0$$

$$4t^2 < y_0^2$$

$$t^2 < \frac{1}{4}y_0^2$$

$$|t| < \frac{|y_0|}{2}$$

4. Verify that both $y_1 = 1 - t$ and $y_2 = -\dfrac{t^2}{4}$ are solutions to the same initial value problem

$$y'(t) = \frac{-t + \sqrt{t^2 + 4y}}{2}, \qquad y(2) = -1.$$

$f(t,y) = \dfrac{-t + \sqrt{t^2+4y}}{2}$

Does the existence of two solutions to the same initial value problem contradict the Existence and Uniqueness Theorem?

$* \quad y_1 = 1 - t \Rightarrow y_1' = -1, \quad \dfrac{-t + \sqrt{t^2 + 4(1-t)}}{2} = \dfrac{-t + \sqrt{t^2 - 4t + 4}}{2}$

$= \dfrac{-t + \sqrt{(t-2)^2}}{2}$

$= \dfrac{-t + t - 2}{2}$

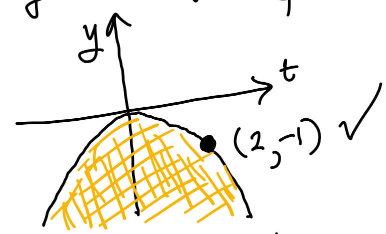
$= -1 \checkmark$ y_1 is a solution

$* \quad y_2 = -\dfrac{t^2}{4}, \quad y_2' = -\dfrac{t}{2}, \quad \dfrac{-t + \sqrt{t^2 + 4\left(-\frac{t^2}{4}\right)}}{2} = \dfrac{-t + \sqrt{0}}{2}$

$= -\dfrac{t}{2} \checkmark$ y_2 is a solution

$* \quad f(t,y) = \dfrac{-t + \sqrt{t^2+4y}}{2}$ is continuous when $t^2 + 4y \geq 0 \Rightarrow y \leq -\dfrac{t^2}{4}$

$*$ $f(t,y)$ is continuous at $(2,-1)$

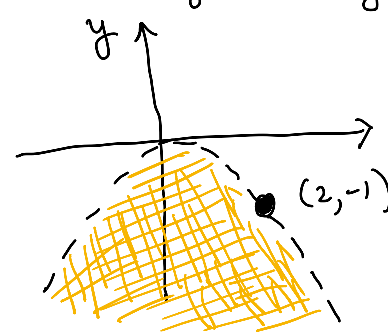


solution exists

$* \quad \dfrac{\partial f}{\partial y}(t,y) = \dfrac{1}{4}(t^2+4y)^{-1/2} \cdot 4$ is continuous when $t^2 + 4y > 0 \Rightarrow y < -t^2/4$

$= \dfrac{1}{\sqrt{t^2+4y}}$

$* \quad \dfrac{\partial f}{\partial y}(t,y)$ is NOT continuous at $(2,-1)$



solution not unique

5. Given the differential equation

$$y' = y^2(9 - y^2)$$

(a) Find the equilibrium solutions.

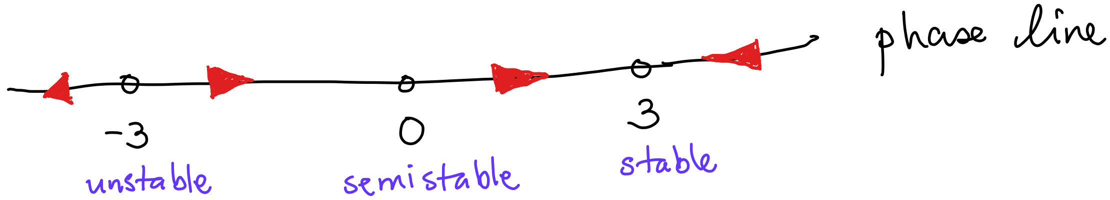
(b) Graph the phase line. Classify each equilibrium solution as either stable, unstable, or semistable

(c) Graph some solutions

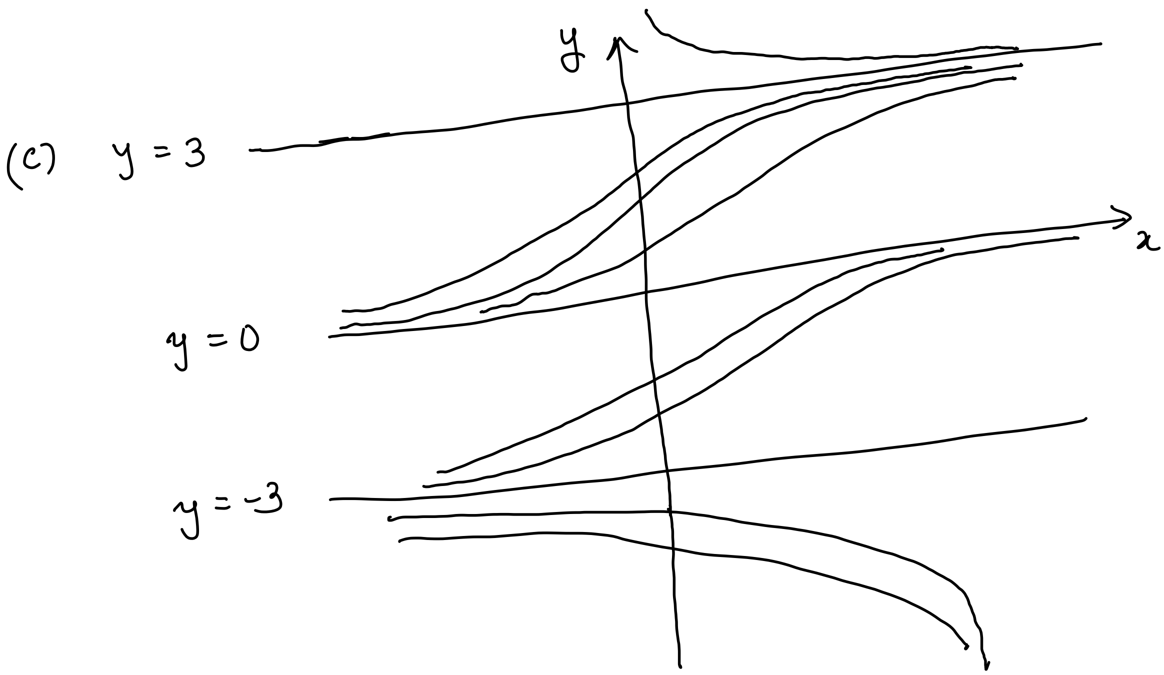
(d) If $y(t)$ is the solution of the equation satisfying the initial condition $y(0) = y_0$ for some $y_0 \in (-\infty, \infty)$, find the limit of $y(t)$ as $t \to \infty$

(a) $f(y) = y^2(9-y^2) = y^2(3-y)(3+y) = 0 \Rightarrow y = -3, 0, 3$

(b)



(c)

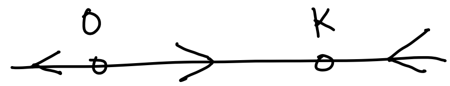


(d)

* $y_0 < -3$, $\lim_{t\to\infty} y(t) = -\infty$	* $y_0 = 0$, $\lim_{t\to\infty} y(t) = 0$
* $y_0 = -3$ $\lim_{t\to\infty} y(t) = -3$	* $0 < y < 3$, $\lim_{t\to\infty} y(t) = 3$
* $-3 < y_0 < 0$ $\lim_{t\to\infty} y(t) = 0$	* $y_0 = 3$, $\lim_{t\to\infty} y(t) = 3$
	* $y_0 > 3$, $\lim_{t\to\infty} y(t) = 3$

6. Suppose a certain population obeys the logistic equation

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right).$$



If $y_0 = K/4$ find the time τ at which the initial population has doubled. Find the value of τ corresponding to $r = 0.05$.

$$\frac{1}{y(1-y/K)}\,dy = \int r\,dt$$

$$\frac{1}{y(1-y/K)} = \frac{A}{y} + \frac{B}{1-y/K}$$

$$1 = A(1-y/K) + By$$

$y=0$: $A = 1$, $y = K$, $B = 1/K$

$$\int \frac{1}{y(1-\frac{y}{K})}\,dy = \int \frac{1}{y}\,dy + \int \frac{1/K}{1-y/K}\,dy = \int r\,dt$$

$$\ln|y| - \cancel{\tfrac{1}{K}}\ln\left|1-y/K\right| \cdot \cancel{K} = rt + C$$

$$\ln\left(\frac{y}{1-y/K}\right) = rt + C$$

$*$ $y > 0$ and $1 - y/K > 0$ when $y_0 = K/4$

$$\frac{y}{1-y/K} = \pm e^{C} \cdot e^{rt} = Ce^{rt}$$

initial conditions $y_0 = K/4$

$$\frac{K/4}{1-(K/4)/K} = K/3 = C$$

$$\frac{y}{1-y/K} = \frac{K}{3}e^{rt}$$

$$y = \frac{K}{3}e^{rt} - \frac{y}{3}e^{rt}$$

$$y\left(1 + \tfrac{1}{3}e^{rt}\right) = \frac{K}{3}e^{rt}$$

$$y = \frac{Ke^{rt}}{3\left(1+\frac{1}{3}e^{rt}\right)} = \frac{Ke^{rt}}{e^{rt}+3}$$

$$\boxed{y = \frac{K}{1+3e^{-rt}}}$$

check: $y(0) = \frac{K}{1+3} = \frac{K}{4}$ ✓

Doubling time: Find t when $y = K/2$

$$\frac{\cancel{K}}{2} = \frac{\cancel{K}}{1+3e^{-rt}}$$

$$\frac{1}{2} = \frac{1}{1+3e^{-rt}}$$

$$1 + 3e^{-rt} = 2$$

$$3e^{-rt} = 1$$

$$e^{-rt} = 1/3$$

$$-rt = \ln\left(\tfrac{1}{3}\right)$$

$$rt = \ln(3)$$

$$\boxed{\tau = \frac{1}{r}\ln(3)}$$

$r = 0.05 \Rightarrow \tau = \frac{1}{0.05}\ln(3)$

$$\tau = 21.97$$

7. Determine if the differential equation is exact. If it is exact, solve it. You may leave your solution in implicit form.

(a)

$$\underbrace{(3x^2y + e^y)}_{M}\,dx + \underbrace{(x^3 + xe^y - 2y)}_{N}\,dy = 0.$$

$M_y = 3x^2 + e^y = N_x$ ✓ exact

$F_x = 3x^2y + e^y$, $F_y = x^3 - xe^y - 2y$

$F(x,y) = \int (3x^2y + e^y)\,dx$

$= x^3y + xe^y + h(y)$

$F_y = x^3 + xe^y + h'(y)$

$h'(y) = -2y$

$h(y) = \int -2y\,dy = -y^2 + C$

$F(x,y) = x^3y + xe^y - y^2 + C$

(b)

$$\underbrace{(3x^2y + 8xy^2)}_{M} + \underbrace{(x^3 + 8x^2y + 12y^2)}_{N}y' = 0.$$

$M_y = 3x^2 + 16xy = N_x$ exact

$F_x = 3x^2y + 8xy^2$

$F(x,y) = \int (3x^2y + 8xy^2)\,dx$

$= x^3y + 4x^2y^2 + h(y)$

$F_y = x^3 + 8x^2y + h'(y)$

$h'(y) = 12y^2$

$h(y) = \int 12y^2\,dy$

$= 4y^3 + C$

$F(x,y) = x^3y + 4x^2y^2 + 4y^3 + C$

8. Consider the differential equation

$$(-xy\sin x + 2y\cos x)\,dx + 2x\cos x\,dy = 0.$$

Show that it is not exact, and that it becomes exact when multiplied by the integrating factor $\mu(x,y) = xy$. Solve. * check if equation is exact

$M = -xy\sin x + 2y\cos x$, $N = 2x\cos x$

$M_y = -x\sin x + 2\cos x$, $N_x = 2\cos x - 2x\sin x$

$M_y \neq N_x$ not exact

* Multiply equation by $\mu(x,y) = xy$ (integrating factor)

$M = -x^2y^2\sin x + 2xy^2\cos x$, $N = 2x^2y\cos x$

$M_y = -2x^2y\sin x + 4xy\cos x$, $N_x = y(4x\cos x - 2x^2\sin x)$

$= -2x^2y\sin x + 4xy\cos x$

now exact

integration by parts

$F_x = -x^2y^2\sin x + 2xy^2\cos x \Rightarrow F(x,y) = \int(-x^2y^2\sin x + 2xy^2\cos x)\,dx$

* let's see if integral in y is easier!

$F_y = 2x^2y\cos x \Rightarrow F(x,y) = \int 2x^2y\cos x\,dy$ } easier integral

$= x^2y^2\cos x + h(x)$

$F_x = 2xy^2\cos x - x^2y^2\sin x + h'(x)$

$= M \Rightarrow h'(x) = 0 \Rightarrow h(x) = C$

$F(x,y) = x^2y^2\cos x + C$