Math 308: Week-in-Review 6

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$y(t) = y_c(t) + y_p(t)$

1. Find the general solution of the following equation

(a)

divide by 2 → standard form

$$2y'' - 3y' + y = (t^2+1)e^t$$

* Find $y_c(t)$ * homogeneous solutions

$2\lambda^2 - 3\lambda + 1 = 0$

$\lambda = \frac{3 \pm \sqrt{3^2 - 4\cdot 2}}{2(2)} = \frac{3 \pm 1}{4} \Rightarrow \lambda = 1, \frac{1}{2}$

$y_1 = e^t, \; y_2 = e^{t/2}$

* Find y_p * particular solution

Variation of parameters: $y_p = u_1 y_1 + u_2 y_2 = u_1 e^t + u_2 e^{t/2}$

$u_1' = -\frac{y_2 r}{W}, \quad u_2' = \frac{y_1 r}{W}$ where $r(t) = \frac{1}{2}(t^2+1)e^t$

Wronskian

$W(t) = \begin{vmatrix} e^t & e^{t/2} \\ e^t & \frac{1}{2}e^{t/2} \end{vmatrix} = \frac{1}{2}e^{3/2 t} - e^{3/2 t}$

$W(t) = -\frac{1}{2}e^{3/2 t} \neq 0$

$u_1' = -\frac{e^{t/2} \cdot \frac{1}{2}(t^2+1)e^t}{-\frac{1}{2}e^{3/2 t}} = t^2 + 1$

$u_1 = \frac{t^3}{3} + t$

$u_2' = \frac{e^t \cdot \frac{1}{2}(t^2+1)\cdot e^t}{-\frac{1}{2}e^{3/2 t}} = -e^{t/2}(t^2+1)$

* Integrate by parts, twice

$u_2 = (-2t^2 + 8t - 18)e^{t/2}$

$y_p = e^t\left(\frac{t^3}{3} + t\right) + e^t(-2t^2 + 8t - 18) = e^t\left(\frac{t^3}{3} - 2t^2 + 9t - 18\right)$

$y(t) = c_1 e^t + c_2 e^{t/2} + e^t\left(\frac{t^3}{3} - 2t^2 + 9t - 18\right)$

(b) Solve the initial value problem

$$3y'' + 4y' + y = (\sin t)\, e^{-t}, \quad y(0) = 1, \quad y'(0) = 0.$$

$y(t) = y_c(t) + y_p(t)$

* Find $y_c(t)$ * $3\lambda^2 + 4\lambda + 1 = 0 \Rightarrow \lambda = \frac{-4 \pm \sqrt{16-12}}{6} = \frac{-4 \pm 2}{6} = -1, -1/3 \Rightarrow y_1 = e^{-t}, \; y_2 = e^{-t/3}$ homogeneous solutions

* Find y_p * particular solution variation of parameters: $y_p = u_1 y_1 + u_2 y_2 = u_1 e^{-t} + u_2 e^{-t/3}$

$W = \begin{vmatrix} e^{-t} & e^{-t/3} \\ -e^{-t} & -\frac{1}{3}e^{-t/3} \end{vmatrix} = -\frac{1}{3}e^{-4/3 t} + e^{-4/3 t}$

$W = 2/3\, e^{-4/3 t}$

$u_1' = -\frac{y_2 r}{W} = -\frac{e^{-t/3} \cdot \frac{1}{3}\sin t \cdot e^{-t}}{2/3\, e^{-4/3 t}} = -\frac{1}{2}\sin t \Rightarrow u_1 = \frac{1}{2}\cos t$

$u_2' = \frac{y_1 r}{W} = \frac{e^{-t}\frac{1}{3}\sin t \cdot e^{-t}}{2/3\, e^{-4/3 t}} = \frac{1}{2}\sin t \cdot e^{-2/3 t}$

* integrate by parts *

$u_2 = -\frac{3}{13}e^{-2t/3}\sin(t) - \frac{9}{26}e^{-2t/3}\cos(t)$

$\Rightarrow y_p = \frac{1}{2}e^{-t}\cos t - \frac{3}{13}e^{-t}\sin t - \frac{9}{26}e^{-t}\cos t = \frac{2}{13}e^{-t}\cos t - \frac{3}{13}e^{-t}\sin t$

General solution

$y(t) = c_1 e^{-t} + c_2 e^{-t/3} + \frac{1}{13}\left(2e^{-t}\cos t - 3e^{-t}\sin t\right)$

* plugin initial conds: $y(0) = 1, y'(0) = 0$ and solve for c_1, c_2

$c_2 = \frac{24}{13}, \; c_1 = -1$

2. Find two linearly independent solutions of $t^2y'' - 2y = 0$ of the form $y(t) = t^r$. Using these solutions, find the general solution of $t^2y'' - 2y = t^2$.

$y(t) = t^r \Rightarrow y'(t) = r t^{r-1}, \; y''(t) = r(r-1)t^{r-2}$

* plug into equation *

for all t

$t^2 \cdot r(r-1) \cdot t^{r-2} - 2t^r = t^r\left[r(r-1) - 2\right] = t^r\left[(r+1)(r-2)\right] = 0$

$r = -1, 2 \Rightarrow y_1 = t^{-1} = \frac{1}{t}$ and $y_2 = t^2$

* standard form eqn: $y'' - \frac{2}{t^2}y = 1$

* variation of parameters: $y_p(t) = u_1y_1 + u_2y_2$

$u_1' = \frac{-y_2 r}{W}, \; u_2' = \frac{y_1 r}{W}$, where $W = \begin{vmatrix} t^{-1} & t^2 \\ -t^{-2} & 2t \end{vmatrix} = 2 + 1 = 3$

$u_1' = -\frac{t^2}{3} \Rightarrow u_1 = -t^3/9, \quad u_2' = \frac{t^{-1}}{3} \Rightarrow u_2 = \frac{1}{3}\ln|t|$

* $y_p(t) = -\frac{1}{9}t^3 \cdot t^{-1} + \frac{1}{3}t^2\ln|t|$ particular solution

$= -\frac{1}{9}t^2 + \frac{1}{3}t^2\ln|t|$

$y_p = t^2\left(\frac{1}{3}\ln|t| - \frac{1}{9}\right)$

* $y(t) = c_1t^{-1} + c_2t^2 + t^2\left(\frac{1}{3}\ln|t| - \frac{1}{9}\right)$

$y(t) = C_1t^{-1} + C_2t^2 + \frac{1}{3}t^2\ln|t|$ where $C_1 = c_1$, $C_2 = c_2 - 1/9$

general solution

3. One solution of $4t^2y'' + 4ty' + (16t^2 - 1)y = 0$, $t > 0$ is $y(t) = t^{-1/2}\cos(2t)$. Find the general solution of $4t^2y'' + 4ty' + (16t^2 - 1)y = 16t^{3/2}$.

* Set $y_1 = t^{-1/2}\cos(2t)$ then find y_2 using Liouville's formula *

$$y_2 = y_1 \int \frac{e^{-\int p(t)dt}}{y_1^2}\,dt$$

where $p(t) = \frac{4t}{4t^2} = \frac{1}{t}$ (standard form)

$$e^{-\int 1/t\,dt} = e^{-\ln t} = \frac{1}{t}$$

$$= t^{-1/2}\cos(2t)\int \frac{t^{-1}}{t^{-1}\cos^2(2t)}\,dt$$

$$= t^{-1/2}\cos(2t)\int \sec^2(2t)\,dt$$

$$= t^{-1/2}\cos(2t)\,\frac{1}{2}\tan(2t)$$

$$= \frac{1}{2}t^{-1/2}\cancel{\cos(2t)}\,\frac{\sin(2t)}{\cancel{\cos(2t)}}$$

$$y_2 = t^{-1/2}\sin(2t)$$

$y_1 = t^{-1/2}\cos(2t)$, $y_2 = t^{-1/2}\sin(2t)$

$$W = y_1y_2' - y_1'y_2$$

$$= \left(t^{-1/2}\cos(2t)\right)\left(-\frac{1}{2}t^{-3/2}\sin(2t) + 2t^{-1/2}\cos(2t)\right)$$

$$- \left(-\frac{1}{2}t^{-3/2}\cos(2t) - 2t^{-1/2}\sin(2t)\right)\left(t^{-1/2}\sin(2t)\right)$$

$$= -\frac{1}{2}t^{-3/2}\cancel{\sin(2t)\cos(2t)} + 2t^{-1}\cos^2(2t)$$

$$+ \frac{1}{2}t^{-3/2}\cancel{\cos(2t)\sin(2t)} + 2t^{-1}\sin^2(2t)$$

$$= 2t^{-1}\left(\underbrace{\cos^2(2t) + \sin^2(2t)}_{1}\right)$$

$$= 2t^{-1}$$ Wronskian

particular solution

$$y_p = u_1y_1 + u_2y_2$$

$$r(t) = \frac{16t^{3/2}}{4t^2} = 4t^{-1/2}$$

$$u_1' = -\frac{y_2 r}{W} = -\frac{t^{-1/2}\sin(2t)\cdot 4t^{-1/2}}{2t^{-1}} = -2\sin(2t)$$

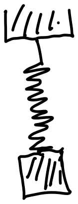
$$u_2' = \frac{y_1 r}{W} = \frac{t^{-1/2}\cos(2t)\,4t^{-1/2}}{2t^{-1}} = 2\cos(2t)$$

$$u_1 = \cos(2t) \qquad u_2 = \sin(2t)$$

$$* \quad y_p = t^{-1/2}\cos^2(2t) + t^{-1/2}\sin^2(2t) = t^{-1/2}\left(\cos^2(2t) + \sin^2(2t)\right) = t^{-1/2}$$

$$y(t) = y_c + y_p = c_1t^{-1/2}\cos(2t) + c_2t^{-1/2}\sin(2t) + t^{-1/2}$$

4. A mass weighing 3 lb stretches a spring 3 in. If the mass is pushed upward, contracting the spring a distance of 1 in, then set in motion with a downward velocity of 2 ft/s, and if there is no damping, find the position u of the mass at any time t. Determine the frequency, period, amplitude, and phase angle of the motion.



downward is positive

$$my'' + cy' + ky = F_{ext}$$

$$k = \frac{3 \text{ lb}}{(3/12)} = 12 \text{ lb/ft}$$

$$y(0) = -1 \text{ in} = -1/12 \text{ ft} \qquad F_{ext} = 0$$

$$y'(0) = 2 \text{ ft/s}$$

$$m = \frac{3}{32} \text{ slug}, \quad c = 0, \text{ no damping}$$

* weight (lb) = mass (slug) × g (ft/s²) *
g (acceleration due to gravity) $\approx 32 \text{ ft/s}^2$

$$\begin{cases} \frac{3}{32}y'' + 12y = 0 \\ y(0) = -1/12 \\ y'(0) = 2 \end{cases}$$

* standard form * $\quad y'' + 128y = 0$

$$r^2 + 128 = 0$$

$$\Rightarrow r = \pm\sqrt{128}\, i = \pm 8\sqrt{2}\, i$$

$$y(t) = c_1 \cos(8\sqrt{2}t) + c_2 \sin(8\sqrt{2}t)$$

$$y'(t) = -8\sqrt{2}c_1 \sin(8\sqrt{2}t) + 8\sqrt{2}c_2 \cos(8\sqrt{2}t)$$

$$y(0) = c_1 = -1/12, \quad y'(0) = 8\sqrt{2}c_2 = 2$$

$$c_2 = \frac{2}{8\sqrt{2}} = \frac{1}{4\sqrt{2}}$$

$$y(t) = -\frac{1}{12}\cos(8\sqrt{2}t) + \frac{1}{4\sqrt{2}}\sin(8\sqrt{2}t)$$

$$= A\cos(8\sqrt{2}t - \varphi)$$

where $A = \sqrt{c_1^2 + c_2^2}$ * amplitude *

$$= \sqrt{\left(-\tfrac{1}{12}\right)^2 + \left(\tfrac{1}{4\sqrt{2}}\right)^2}$$

$$= \sqrt{\frac{1}{144} + \frac{1}{32}} = \sqrt{\frac{11}{288}}$$

* frequency = $\omega = 8\sqrt{2}$ rad/sec

* period = $T = \frac{2\pi}{\omega} = \frac{\pi}{4\sqrt{2}}$ sec

$$\tan\varphi = \frac{c_2}{c_1} = \frac{\frac{1}{4\sqrt{2}}}{-1/12} = -\frac{3}{\sqrt{2}}$$

$$\varphi = \pi + \tan^{-1}\left(-\frac{3}{\sqrt{2}}\right)$$ * phase angle *

* because we are in Quadrant II *

5. A spring is stretched 10 cm by a force of 3 N. A mass of 2kg is hung from the spring and is also attached to a viscous damper that exerts a force of 3 N when the velocity of the mass is 5 m/s. If the mass is pulled down 5cm below its equilibrium position and given an initial velocity of 10 cm/s, determine its position u at any time. Find the quasifrequency of the motion.

$$my'' + cy' + ky = F_{ext}, \quad F_{ext} = 0$$

* $m = 2\text{kg}$
* $k = \frac{3}{0.1} = 30 \text{ N/m}$
* $c = 3/5 \text{ N/(m/s)} = \frac{Ns}{m}$ (damping)

* Initial value problem *

$$\begin{cases} 2y'' + \frac{3}{5}y' + 30y = 0 \\ y(0) = 0.05\text{m} \\ y'(0) = 0.1 \text{ m/s} \end{cases} \Rightarrow 10y'' + 3y' + 150y = 0$$

$$10\lambda^2 + 3\lambda + 150 = 0$$

$$\lambda = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 10 \cdot 150}}{20} = \frac{-3 \pm \sqrt{9 - 6000}}{20}$$

$$\lambda = -\frac{3}{20} \pm \frac{\sqrt{5991}}{20} i$$

$$\boxed{\text{* Quasifrequency } \frac{\sqrt{5991}}{20} \text{ rad/sec}}$$

$$y(t) = c_1 e^{-\frac{3}{20}t} \cos\left(\frac{\sqrt{5991}}{20}t\right) + c_2 e^{-\frac{3}{20}t} \sin\left(\frac{\sqrt{5991}}{20}t\right)$$

$$y'(t) = -\frac{3}{20}c_1 e^{-\frac{3}{20}t} \cos\left(\frac{\sqrt{5991}}{20}\right) - \frac{\sqrt{5991}}{20} c_1 e^{-\frac{3}{20}t} \sin\left(\frac{\sqrt{5991}}{20}t\right) - \frac{3}{20} c_2 e^{-3/20 t} \sin\left(\frac{\sqrt{5991}}{20}t\right) + \frac{\sqrt{5991}}{20} c_2 e^{-\frac{3}{20}t} \cos\left(\frac{\sqrt{5991}}{20}t\right)$$

$$y(0) = \boxed{c_1 = 0.05}$$

$$y'(0) = -\frac{3}{20}c_1 + \frac{\sqrt{5991}}{20}c_2 = -\frac{3}{20}(0.05) + \frac{\sqrt{5991}}{20}c_2 = 0.1$$

$$\Rightarrow \boxed{c_2} = \frac{\frac{2}{\cancel{20}} + \frac{3}{\cancel{20}}(0.05)}{\sqrt{5991}/\cancel{20}} = \boxed{\frac{2.15}{\sqrt{5991}}}$$

$$\boxed{y(t) = 0.05 e^{-\frac{3}{20}t} \cos\left(\frac{\sqrt{5991}}{20}t\right) + \frac{2.15}{\sqrt{5991}} e^{-\frac{3}{20}t} \sin\left(\frac{\sqrt{5991}}{20}t\right)}$$

6. A spring is stretched 6 in by a mass that weighs 8 lb. The mass is attached to a dashpot mechanism that has a damping constant of 0.25 lb s/ft. and is acted on by an external force of $4\cos(2t)$ lb.

   (a) Find the steady-state response of this system.

   (b) If the given mass is replaced by a mass m, determine the value of m for which the amplitude of the steady state response is maximum.

   (c) If the mass is the same as in the problem, determine the value of ω of the frequency of the external force $4\cos(\omega t)$ at which "practical resonance" occurs, i.e. the amplitude of the steady-state response is maximized.

$$m = \frac{8\text{ lb}}{32\text{ ft}/s^2} = \frac{1}{4}\text{ slug}$$

$$F_{ext} = 4\cos(2t)\text{ lb}$$

$$k = \frac{8\text{ lb}}{(6/12)\text{ ft}} = 16\text{ lb/ft}$$

$$c = 0.25\text{ lbs/ft}$$

$$my'' + cy' + ky = F_0\cos(\omega t)$$

$$\frac{1}{4}y'' + \frac{1}{4}y' + 16y = 4\cos(2t)$$

$$y = \underbrace{y_c}_{\text{transient}} + y_p \rightarrow \text{steady state}$$

* Find homogeneous soln *

$$\lambda = \frac{-\frac{1}{4} \pm \sqrt{\frac{1}{16} - 16}}{2(1/4)} = \frac{-\frac{1}{4} \pm \frac{\sqrt{255}}{4}i}{1/2}$$

$$\lambda = -\frac{1}{2} \pm \frac{\sqrt{255}}{2}i$$

$$* \; y_c(t) = c_1 e^{-t/2}\cos\left(\frac{\sqrt{255}}{2}t\right) + c_2 e^{-t/2}\sin\left(\frac{\sqrt{255}}{2}t\right)$$

$$y = y_p = A\cos(\omega t) + B\sin(\omega t) \quad \text{* no resonance *}$$

$$my'' + cy' + ky = F_0\cos(\omega t)$$

$$m(-A\omega^2\cos\omega t - B\omega^2\sin(\omega t)) + c(-A\omega\sin\omega t + B\omega\cos\omega t) + k(A\cos\omega t + B\sin\omega t) = F_0\cos\omega t$$

* $\cos\omega t$ *: $(k - m\omega^2)A + (c\omega)B = F_0$

* $\sin\omega t$ *: $(-c\omega)A + (k - m\omega^2)B = 0$

$$B = \frac{c\omega A}{k - m\omega^2}$$

$$(k - m\omega^2)A + \frac{(c\omega)^2 A}{k - m\omega^2} = F_0$$

$$A\left[(k - m\omega^2) + \frac{(c\omega)^2}{k - m\omega^2}\right] = F_0 \Rightarrow A = \frac{F_0(k - m\omega^2)}{(k - m\omega^2) + (c\omega)^2}$$

$$B = \frac{F_0(c\omega)}{(k - m\omega^2) + (c\omega)^2}$$

$$\boxed{y_p = \frac{F_0(k - m\omega^2)}{(k - m\omega^2) + (c\omega)^2}\cos\omega t + \frac{F_0(c\omega)}{(k - m\omega^2) + (c\omega)^2}\sin\omega t}$$

* steady state solution *

(a) $F_0 = 4,\ \omega = 2,\ k = 16,\ m = 1/4,\ c = 1/4.$

Plug in these values to get y_p

(b) $R = \sqrt{A^2 + B^2}$

$$= \sqrt{\frac{F_0^2 (k - m\omega^2)^2 + F_0^2 (c\omega)^2}{[(k - m\omega^2)^2 + (c\omega)^2]^2}}$$

$$= \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$ * amplitude *

R is maximized exactly when $f(m) = (k - m\omega^2)^2 + (c\omega)^2$ is minimized

$f(m) = (16 - 4m)^2 + \left(\frac{1}{2}\right)^2 = (16 - 4m)^2 + \frac{1}{4}$

$f(m)$ is minimum when $16 - 4m = 0 \Rightarrow \boxed{m = 4 \text{ slug}}$

(c) minimize $g(\omega) = (k - m\omega^2)^2 + (c\omega)^2$

$$= \left(16 - \frac{1}{4}\omega^2\right)^2 + \left(\frac{1}{4}\omega\right)^2$$

$$g'(\omega) = 0 \Rightarrow 2\left(16 - \frac{1}{4}\omega^2\right)\left(-\frac{1}{2}\omega\right) + 2\left(\frac{1}{4}\omega\right)\left(\frac{1}{4}\right)$$

$$= -\omega\left(16 - \frac{1}{4}\omega^2\right) + \frac{1}{8}\omega$$

$$\frac{1}{8}\cancel{\omega} = \cancel{\omega}\left(16 - \frac{1}{4}\omega^2\right) \Rightarrow \frac{1}{8} = 16 - \frac{1}{4}\omega^2$$

$$\frac{1}{4}\omega^2 = 16 - \frac{1}{8} \Rightarrow \omega^2 = 64 - \frac{1}{2}$$

$$\omega_{res} = \sqrt{64 - 1/2} = \sqrt{63.5}\ \frac{\text{rad}}{\text{sec}}$$

(which is slightly less than the natural frequency of $\omega_0 = \sqrt{\frac{K}{m}} = \sqrt{\frac{16}{1/4}} = \sqrt{64}$ rad/sec)