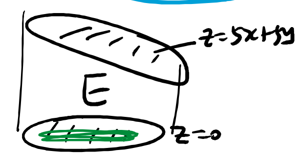
Note ♯6: Sections 15.6-15.8

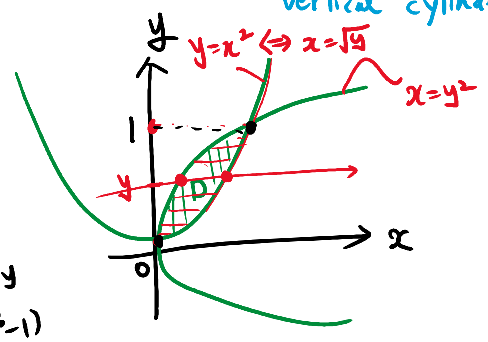
Problem 1. Let E be the region bounded by $y = x^2$ and $x = y^2$ and $z = 0$ and $z = 5x + 5y$. Compute $\iiint_E 4xy\, dV$. Just set up without evaluation.

$$\boxed{\int_0^1 \int_{y^2}^{\sqrt{y}} \int_0^{5x+5y} 4xy \, dz\, dx\, dy}$$

$= \cdots$



Vertical cylinder



$y = (y^2)^2$

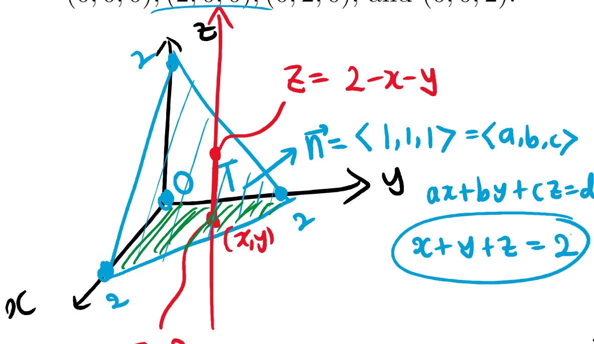
$\Leftrightarrow y = y^4$

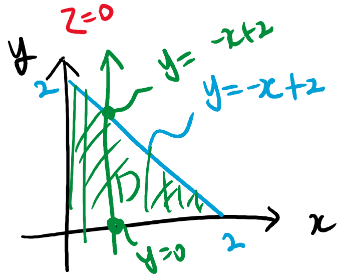
$\Leftrightarrow 0 = y^4 - y$

$\Leftrightarrow 0 = y(y^3 - 1)$

$\Leftrightarrow y = 0,\ y = 1$

Problem 2. Evaluate the triple integral $\iiint_T y^2 dV$, where T is the solid tetrahedron with vertices $(0, 0, 0)$, $(2, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$.





$$\int_0^2 \left(\int_0^{-x+2} \left(\int_0^{2-x-y} y^2 \, dz \right) dy \right) dx$$

$$= \int_0^2 \int_0^{-x+2} \underbrace{y^2(2-x-y)}_{(2-x)y^2 - y^3} \, dy\, dx$$

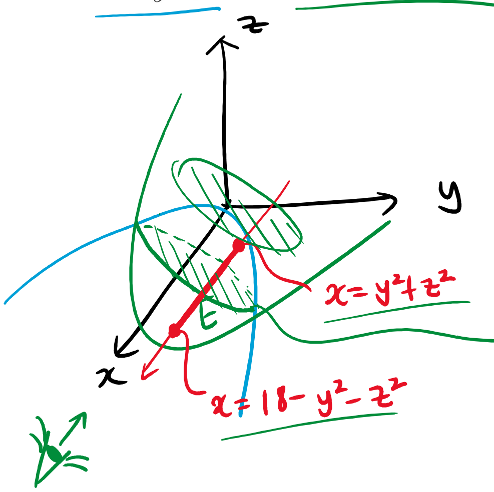
$$= \int_0^2 \left[\frac{2-x}{3} y^3 - \frac{y^4}{4} \right]_0^{-x+2} dx$$

$$= \int_0^2 \frac{1}{3}(-x+2)^4 - \frac{(-x+2)^4}{4} dx$$

$$= \int_0^2 \frac{1}{12}(x-2)^4 dx = \frac{1}{12} \cdot \left[\frac{(x-2)^5}{5} \right]_0^2 = \frac{1}{12}\left(-\frac{(-2)^5}{5} \right)$$

$$= \frac{32}{60} = \boxed{\frac{8}{15}}$$

Problem 3. Evaluate $\iiint_E \sqrt{y^2+z^2}\,dV$, where E is the solid between the elliptic paraboloids $x = y^2 + z^2$ and $x = 18 - y^2 - z^2$.



$$\iint\left(\int_{y^2+z^2}^{18-y^2-z^2}\sqrt{y^2+z^2}\;dx\right)dy\,dz$$

$y^2+z^2 = 18-y^2-z^2$

$\Leftrightarrow 2y^2+2z^2=18$

$\Leftrightarrow y^2+z^2=18$

$[r\,x]_{x=r^2}^{18-r^2} = r((18-r^2)-r^2)$

$$= \int_0^{2\pi}\int_0^{\sqrt{18}}\int_{r^2}^{18-r^2} r\;dx\,(r\,dr\,d\theta)$$

$y = r\cos\theta$

$z = r\sin\theta$

$y^2+z^2=r^2$

$r(18-r^2)$

$$= \int_0^{2\pi}\int_0^{\sqrt{18}} r^2(18-r^2)\,dr\,d\theta$$

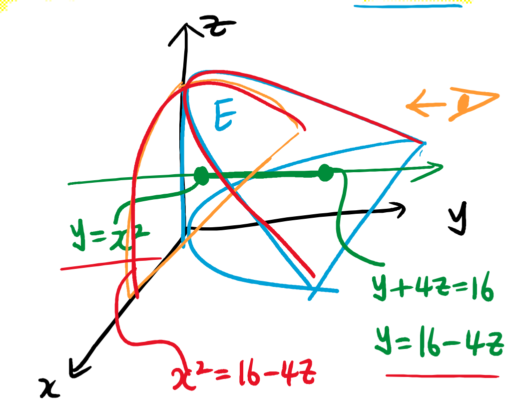
$$= (2\pi)\left[\frac{r^3}{3}\cdot 18 - \frac{r^5}{5}\right]_{r=0}^{\sqrt{18}}$$

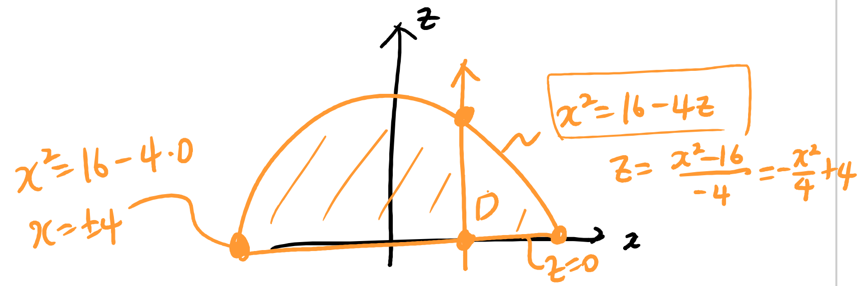
$$= (2\pi)\left(\frac{18^{\frac{5}{2}}}{3} - \frac{18^5}{5}\right) = \boxed{(2\pi)(18^5)\left(\frac{2}{15}\right)}$$

Problem 4. Express $\iiint_E f(x,y,z)dV$ in the order $dy\,dz\,dx$ if E is the solid bounded by $y = x^2$, $z = 0$, $y + 4z = 16$.

$\Leftrightarrow z = \frac{16-y}{4} = 4 - \frac{1}{4}y$

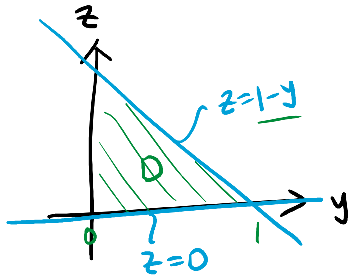
$$\boxed{\int_{-4}^{4}\int_{0}^{-\frac{x^2}{4}+4}\int_{x^2}^{16-4z} f(x,y,z)\;dy\,dz\,dx}$$

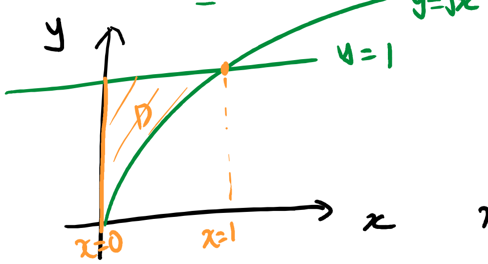


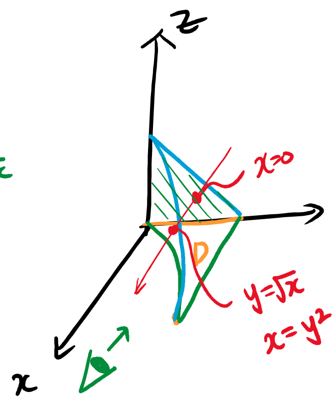


Problem 5. Rewrite the integral as an equivalent iterated integral in the five other orders.

$$\int_{0}^{1}\int_{\sqrt{x}}^{1}\int_{0}^{1-y} f(x,y,z)\,dz\,dy\,dx$$







$$\int_{0}^{1}\int_{0}^{1-y}\int_{0}^{y^2} f\,dx\,dz\,dy$$

⋮

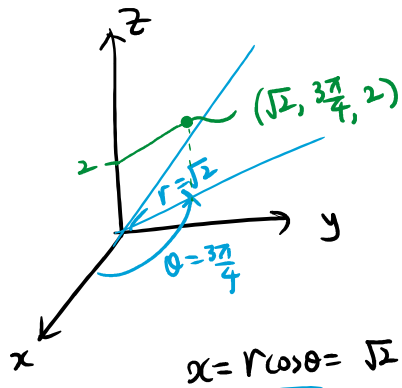
4 more integrals

Problem 6. (a) Plot the point whose cylindrical coordinates are $(\sqrt{2}, 3\pi/4, 2)$. Then find the rectangular coordinates of the point.

rectangular (x, y, z) cylindrical (r, θ, z)

polar

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ x^2 + y^2 = r^2 \\ \frac{y}{x} = \tan\theta \end{cases}$$



$$x = r\cos\theta = \sqrt{2}\cos\left(\frac{3\pi}{4}\right) = \sqrt{2}\left(-\frac{1}{\sqrt{2}}\right) = -1$$

$$y = r\sin\theta = \sqrt{2}\sin\left(\frac{3\pi}{4}\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 1$$

$$\boxed{(x, y, z) = (-1, 1, 2)}$$

(b) Change from rectangular to cylindrical coordinates of the point $(-2, 2\sqrt{3}, 3)$.

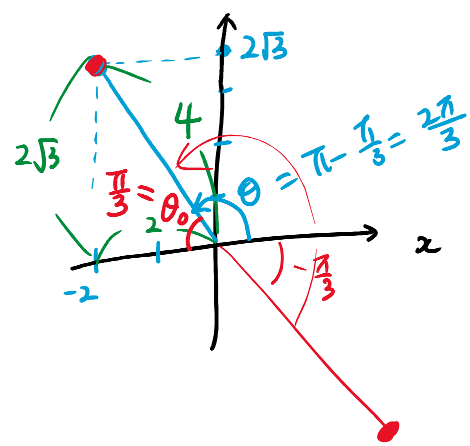
~ 1.7

$x \quad y \quad z$

$$\boxed{r} = \boxed{\sqrt{x^2 + y^2}} = \sqrt{2^2 + (2\sqrt{3})^2} = \sqrt{4 + 12}$$

$$= \sqrt{16} = \boxed{4}$$

$$\boxed{\theta} = \boxed{\frac{2\pi}{3}}$$

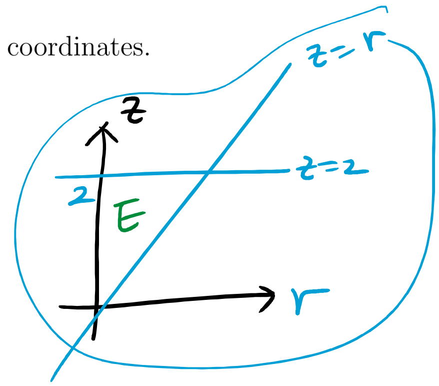


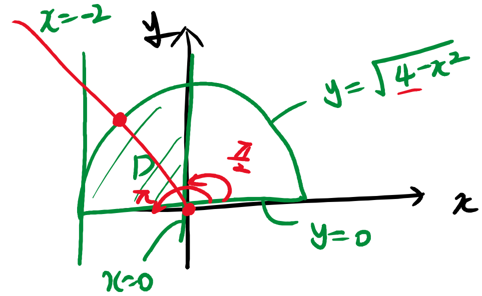
$$\frac{y}{x} = \frac{2\sqrt{3}}{-2} = -\sqrt{3} = \tan\theta$$

$$\theta = \left(\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}\right) + \pi$$

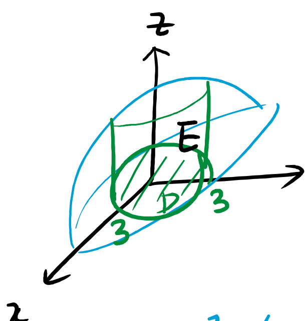
Problem 7. Convert $\int_{-2}^{0}\int_{0}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} xz\,dz\,dy\,dx$ to cylindrical coordinates.

$$\boxed{= \int_{\frac{\pi}{2}}^{\pi}\int_{0}^{2}\int_{r}^{2} \underbrace{(r\cos\theta)\, z\, (dz)(r\,dr\,d\theta)}_{r^2 z\cos\theta\, dz\, dr\, d\theta}}$$





Problem 8. Find the volume of the sold that is above the xy plane, below the ellipsoid $4x^2 + 4y^2 + z^2 = 64$ but inside the cylinder $x^2 + y^2 = 9$.



$$V = \iiint_E 1\, dV$$

$$= \int_0^{2\pi}\int_0^3 \left(\int_0^{\sqrt{64-4r^2}} 1\, dz\right)(r\,dr\,d\theta)$$

$$z^2 = 64 - 4(x^2+y^2)$$

$$z = \sqrt{64-4(x^2+y^2)}$$

$$= \sqrt{64-4r^2}$$

$$= \int_0^{2\pi} d\theta \int_0^3 \sqrt{64-4r^2}\, r\, dr$$

$u = 64-4r^2$

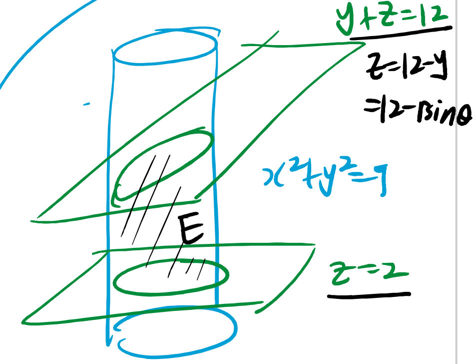
$du = -8r\,dr$

$$= (2\pi)\left[-\frac{1}{8}\frac{2}{3}\left(64-4r^2\right)^{\frac{3}{2}}\right]_0^3 = \boxed{(2\pi)\left(-\frac{1}{12}\left(28^{\frac{3}{2}} - (64)^{\frac{3}{2}}\right)\right)}$$

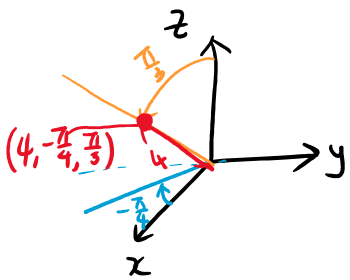
Problem 9. Find the volume of the solid that is enclosed by the cylinder $x^2 + y^2 = 9$ and the planes $y + z = 12$ and $z = 2$.

$$\boxed{V = \int_0^{2\pi} \int_0^{3} \int_2^{12 - r\sin\theta} 1 \, dz \, (r\,dr\,d\theta)}$$

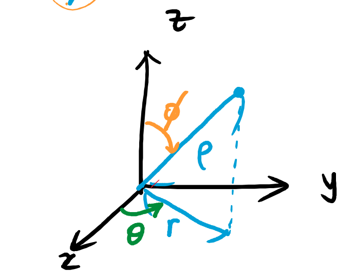
disk of
$r=3$

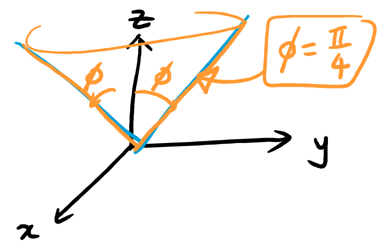


Problem 10. (a) Plot the point whose spherical coordinates are $(\underbrace{4}_{\rho}, \underbrace{-\pi/4}_{\theta}, \underbrace{\pi/3}_{\phi})$. Then find the rectangular coordinates of the point.



$$\begin{cases} \rho = \sqrt{x^2+y^2+z^2} \\ r = \sqrt{x^2+y^2} \\ z = \rho\cos\phi \\ r = \rho\sin\phi \\ x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \end{cases}$$





$$z = \rho\cos\phi = 4\cos\left(\frac{\pi}{3}\right) = 4\cdot\frac{1}{2} = 2$$

$$r = \rho\sin\phi = 4\sin\left(\frac{\pi}{3}\right) = 4\cdot\frac{\sqrt{3}}{2} = 2\sqrt{3}$$

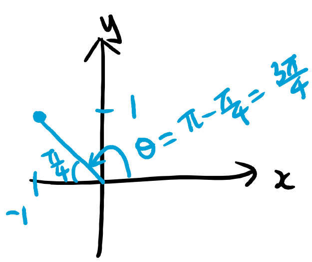
$$x = r\cos\theta = (2\sqrt{3})\cos\left(-\frac{\pi}{4}\right) = (2\sqrt{3})\cdot\frac{\sqrt{2}}{2} = \sqrt{6}$$

$$y = r\sin\theta = (2\sqrt{3})\sin\left(-\frac{\pi}{4}\right) = (2\sqrt{3})\left(-\frac{\sqrt{2}}{2}\right) = -\sqrt{6}$$

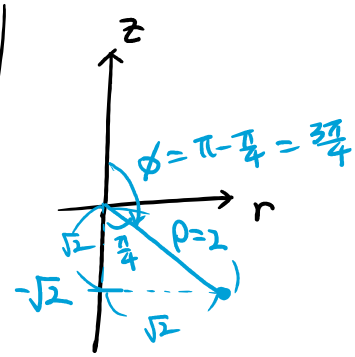
$$\boxed{(\sqrt{6}, -\sqrt{6}, 2)}$$

(b) Change from rectangular to spherical coordinates of the point $(\overset{x}{-1}, \overset{y}{1}, \overset{z}{-\sqrt{2}})$.

$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{1^2+1^2+\sqrt{2}^2} = \sqrt{1+1+2} = \sqrt{4} = 2$$



$$\theta = \frac{3\pi}{4}$$



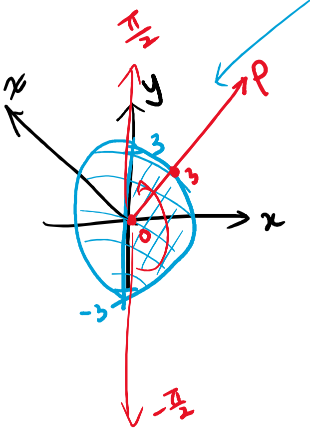
$$\phi = \frac{3\pi}{4}$$

$$\boxed{\left(2, \frac{3\pi}{4}, \frac{3\pi}{4}\right)}$$

$9-y^2=x^2$
$9=x^2+y^2$

$9-x^2-y^2=z^2 \iff x^2+y^2+z^2=9$

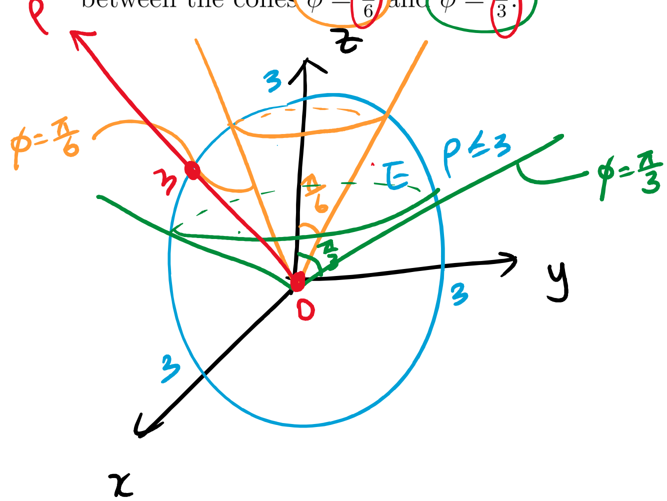
Problem 11. Convert $\int_{-3}^{3}\int_{0}^{\sqrt{9-y^2}}\int_{0}^{\sqrt{9-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$ to spherical coordinates.



$$= \int_0^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^3 (\rho\cos\phi)\,\rho\,\boxed{\rho^2\sin\phi}\,d\rho\,d\theta\,d\phi$$

$dx\,dy\,dz$
$\to r\,dr\,d\theta\,dz$
$\to (\rho^2\sin\phi)\,d\rho\,d\theta\,d\phi$

Problem 12. Using spherical coordinates, find the volume of the part of the ball $\rho \le 3$ that lies between the cones $\phi = \frac{\pi}{6}$ and $\phi = \frac{\pi}{3}$.



$$V=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\int_0^{2\pi}\int_0^3 1\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

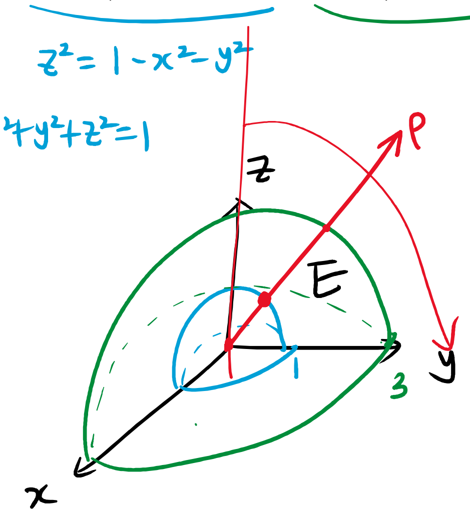
$$=\left(\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sin\phi\,d\phi\right)\left(\int_0^{2\pi}1\,d\theta\right)\left(\int_0^3\rho^2\,d\rho\right)$$

$$=\cdots$$

Problem 13. Evaluate $\iiint_E z^2 dV$, where E is bounded by the xy-plane and the hemispheres $0 \leq \underbrace{z = \sqrt{1 - x^2 - y^2}}$ and $\underbrace{z = \sqrt{9 - x^2 - y^2}} \geq 0$.

$z^2 = 1 - x^2 - y^2$

$x^2 + y^2 + z^2 = 1$



$$= \boxed{\int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_1^3 (\rho \cos\phi)^2 (\rho^2 \sin\phi)\, d\rho\, d\theta\, d\phi}$$