1. Given the following differential equations and their corresponding direction field, determine the behavior as \( t \) increases.

![Fig. 1: \( y'(t) = 2y(t) - 3 \)](image)

\[
y'(t) = 2y(t) - 3
\]

Equilibrium:
- \( y(0) = \frac{3}{2} \Rightarrow y(t) = \frac{3}{2} \)
- \( y(0) > \frac{3}{2} \Rightarrow \lim_{t \to \infty} y(t) = \infty \) as \( t \) increases
- \( y(0) < \frac{3}{2} \Rightarrow \lim_{t \to \infty} y(t) = -\infty \)

![Fig. 2: \( y'(t) = y(t)(2 - y(t)) \)](image)

\[
y'(t) = y(t)(2 - y(t))
\]

Equilibrium:
- \( y(0) = 0 \Rightarrow y(t) = 0 \)
- \( y(0) = 2 \Rightarrow y(t) = 2 \)

\( y(t) \to \infty \) as \( t \) increases

\( 0 \leq y(t) \leq 2 \Rightarrow y(t) \to 2 \)

\( y(t) < 0 \Rightarrow y(t) \to -\infty \)

![Fig. 3: \( y'(t) = -1 - y(t) \)](image)

\[
y'(t) = -1 - y(t)
\]

Non-autonomous

Can we find the equation of the linear solution of the last differential equation?

\[
y = at + b
\]

\[
y' = t - 1 - y
\]

\[
a = t - 1 - (at + b)
\]

\[
a = (1-a)t - 1 - b \quad \text{for all} \ t
\]

\[
1-a=0 \quad \Rightarrow \quad a=1
\]

\[
-1-b=a=1 \quad \Rightarrow \quad b=-2
\]

\[
y = t - 2
\]

\[
y = at + b \Rightarrow y' = a \Rightarrow y'' = 0
\]

\[
y' = t - 1 - y
\]

\[
y'' = 1 - y
\]

\[
y' = a
\]

\[
y = at + b
\]

\[
y' = t - 1 - y
\]

\[
0 = 1 - y' \Rightarrow a=1
\]
2. Given the differential equation \( \frac{dy}{dt} = ty - 1. \)

(a) What is the slope of the graph of the solutions at \((0, 1)\), at the point \((1, 1)\), at the point \((3, -1)\), at the point \((0, 0)\)?

\[ \begin{align*}
(0,1), \text{ slope} &= \left. \frac{dy}{dt} \right|_{t=0} = 0 (1) - 1 = -1 \\
(1,1), \text{ slope} &= 1 (1) - 1 = 0 \\
(3,-1), \text{ slope} &= 3 (-1) - 1 = -4 \\
(0,0), \text{ slope} &= 0 (0) - 1 = -1
\end{align*} \]

(b) Find all the points where the tangents to the solution curves are horizontal.

\[ \frac{dy}{dt} = 0 \implies ty - 1 = 0 \implies y = \frac{1}{t} \]

Critical points of all solution curves \( = \{t, \left( a_1 \right) : t \in \mathbb{R}\backslash\{0\} \} \)

(c) Describe the nature of the critical points.

\[ y'' > 0 \quad \frac{dy}{dt} = ty - 1 \]
\[ y'' = y + ty' \text{ by product rule} \]

@ critical points \( y'' > 0 \text{ and } y > 0 \), \( y'' < 0 \text{ if } y < 0 \)

3. The instantaneous rate of change of the temperature \( T \) of coffee at time \( t \) is proportional to the difference between the temperature \( M \) of the air and the temperature \( T \) at time \( t \).

(a) Find the mathematical model for the problem.

Newton's law of cooling

\[ \frac{dT}{dt} = k(M - T) \]

Units: \( \frac{1}{\text{Sec}} \)
(b) Given that the room temperature is 75° and $k = 0.08$, find the solutions to the differential equation.

$$\frac{dT}{T-75} = 0.08 \, dt$$

Linear

$$-\ln |T-75| = 0.08 \, t + C_1$$

$C$ : any real #

(c) The initial temperature of the coffee is 200°F. Find the solution to the problem.

$$T(0) = 200 = 75 + C \, e^0 \Rightarrow C = 125°$$

$$T = 75 + 125 \, e^{-0.08 \, t} \quad t \to \infty$$

4. Your swimming pool containing 60,000 gal of water has been contaminated by 5 kg of a non toxic dye that leaves a swimmer’s skin an unattractive green. The pool’s filtering system can take water from the pool, remove the dye, and return the water to the pool at a flow rate of 200 gal/min.

(a) Write down the initial value problem for the filtering process; let $q(t)$ be the amount of dye in the pool at any time $t$.

$$\frac{dq}{dt} = -\frac{200}{60,000} \, q(t)$$

$q(0) = 5$ kg

(b) Solve the problem.

$$\frac{dq}{dt} = -\frac{1}{300} \, q(t) \Rightarrow \int \frac{dq}{q} = \int -\frac{1}{300} \, dt$$

$$\ln q = -\frac{1}{300} \, t + C$$

$t = 0, q = 5$ \Rightarrow $\ln 5 = C$ \Rightarrow $\ln q = -\frac{1}{300} \, t + \ln 5$

$q(t) = 5e^{-\frac{t}{300}}$
(c) You have invited several dozen friends to a pool party that is scheduled to begin in 4 hours. You have also determined that the effect of the dye is imperceptible if its concentration is less than 0.02 g/gal. Is your filtering system capable of reducing the dye concentration to this level within 4 hours?

\[ Q(2,400) = 5 e^{-\frac{240}{300}} \approx 1.72 \text{ kg} \]

\[ \frac{1.72}{60,000} = 2.87 \times 10^{-5} \text{ kg/gal} < 0.02 \text{ g/gal} = 2 \times 10^{-5} \text{ kg/gal} \]

It is not enough time.

5. The direction field for the differential equation

\[ x'(t) = \frac{2tx(t)}{1 + x(t)} \]

is given below.

Sketch the graph of the solutions to the initial value problems

(a) \( x(0) = 1 \)
(b) \( x(0) = -2 \)
(c) \( x(0) = -0.5 \)
6. Match the direction field to the differential equations

\[ a) \quad y' = y - 2 \quad b) \quad y' = \frac{2 - y}{y - 2} \quad c) \quad y' = 2 + y \]
\[ d) \quad y' = -2 - y \quad e) \quad y' = (y - 2)^2 \quad f) \quad y' = (y + 2)^2 \]
7. Given the following differential equations, classify each as an ordinary differential equation, partial differential equation, give the order. If the equation is an ordinary differential equation, say whether the equation is linear or non linear.

(a) \( \frac{dy}{dx} = 3y + x^2 \). \( \text{ODE} \) \( 1 \) \( L \)

(b) \( \frac{d^4 y}{dx^4} + y = x(x - 1) \). \( \text{ODE} \) \( 4 \) \( L \)

(c) \( \frac{\partial N}{\partial t} = \frac{\partial^2 N}{\partial r^2} + \frac{\partial N}{\partial r} + 6N \). \( \text{PDE} \) \( 2 \) \( L \)

(d) \( \frac{dy}{dt} = x^2 - t \). \( \text{ODE} \) \( 1 \) \( NL \)

(e) \( (1 + y^2) y'' + ty' + y = e^t \). \( \text{ODE} \) \( 2 \) \( NL \)

8. (a) Show that \( f(x) = (x^2 + Ax + B)e^{-x} \) is solution to

\[
\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + y = 2e^{-x}
\]

for all real numbers \( A \) and \( B \).

\[
\begin{align*}
y'' &= (2x + A)e^{-x} + (x^2 + Ax + B)(-e^{-x}) \\
y' &= 2e^{-x} + (2x + A)(-e^{-x}) + (x^2 + Ax + B)(e^{-x}) \\
y &= \frac{2e^{-x} - 2(2x + A)e^{-x} + (x^2 + Ax + B)e^{-x}}{2(2x + A)e^{-x} - 2(x^2 + Ax + B)e^{-x}} \\
&= 2e^{-x}
\end{align*}
\]

(b) Find a solution that satisfies the initial condition \( y(0) = 3 \) and \( y'(0) = 1 \).

\[
y(0) = 3 \Rightarrow B = 3
\]

\[
y'(0) = 1 \Rightarrow (6 + A)(1) + (0 + 0 + 3)(-1) = 1
\]

\[
\Rightarrow A - 3 = 1 \Rightarrow A = 4
\]

\[
y(x) = (x^2 + 4x + 3)e^{-x}
\]
9. Determine for which values of $r$ the function $y = t^r$ is a solution of the differential equation

$$t^2 y'' - 4 t y' + 4 t^2 = 0, \quad t > 0.$$ 

We have $y' = r t^{r-1}$ and $y'' = r(r-1) t^{r-2}$. Substituting into the differential equation gives

$$t^2 [r(r-1) t^{r-2}] - 4 t r t^{r-1} + 4 t^r = 0,$$

which simplifies to

$$r(r-1) t^r - 4 r t^r + 4 t^r = 0.$$

This is the characteristic equation.

$$r(r-1) - 4 r + 4 = 0$$

Solving for $r$ gives $r = 1, 4$.

So, $y = t$ and $y = 4$ are solutions for $r = 1, 4$.

10. For which values of $r$ is the function $y = (x-1)e^{-rx}$ solution to $y'' - 6y' + 9y = 0$?

We have $y' = e^{-rx} + (x-1)(-re^{-rx}) = (1+r-rx)e^{-rx}$ and $y'' = -r e^{-rx} + (1+r-rx)(-re^{-rx}) = (r^2 - r^2 - 2r) e^{-rx}$.

Substituting into the differential equation gives

$$y'' - 6y' + 9y = (r^2 - 2r - 6(1+r-rx) + 9(x-1)) e^{-rx} = 0$$

for all $x$.

Solving $r^2 + 6r + 9 = 0$ gives $r = -3$.

Therefore, $y = (x-1)e^{3x}$ is a solution for $r = -3$. 

$$(-3)^3 + 8(-3) + 15 = 0$$

So, $y = (x-1)e^{3x}$ is indeed a solution for $r = -3$. 

11. In curling, the player has to slide a stone of mass $m$ on a smooth surface with friction coefficient $\mu$. If the air friction is proportional to the velocity of the stone with a drag coefficient $\gamma$, find the stopping time and the stopping distance of the stone if the initial velocity is $v_0$.

\[
F = m \ a
\]

Initial Value Problem (IVP)

\[
\begin{align*}
- \mu mg - \gamma v &= m \ a = m \ \frac{dv}{dt} \\
\quad v(0) &= v_0
\end{align*}
\]

the stopping time $T$ $\Rightarrow$ $\quad v(T) = 0$

\[
\frac{dv}{dt} = -\mu g - \frac{\gamma}{m} \ v = -(\mu g + \frac{\gamma}{m} \ v)
\]

\[
\int_{v_0}^{0} \frac{dv}{\frac{\gamma}{m} v + mg} = -\int_{0}^{T} dt = -T
\]

\[
T = \left[ \ln \left( \frac{\frac{\gamma}{m} v_0 + mg}{mg} \right) \right]_{v=0}^{v_0}
\]

$\Rightarrow T = \frac{m}{\gamma} \left[ \ln \left( \frac{\frac{\gamma}{m} v_0 + mg}{mg} \right) - \ln \left( mg \right) \right]$

\[
T = \frac{m}{\gamma} \ln \left( \frac{\frac{\gamma}{m} v_0 + mg}{mg} \right) = \frac{m}{\gamma} \ln \left( 1 + \frac{\gamma v_0}{\mu mg} \right)
\]
11. In curling, the player has to slide a stone of mass \( m \) on a smooth surface with friction coefficient \( \mu \). If the air friction is proportional to the velocity of the stone with a drag coefficient of \( \gamma \), find the stopping time and the stopping distance of the stone if the initial velocity is \( v_0 \).

\[
\text{Stopping Distance} : \text{ change variable (Chain Rule)}
\]

\[
a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v \frac{dv}{dx}
\]

\[
\begin{align*}
\{ & m \, v \frac{dv}{dx} = -\mu mg - \gamma v \\
& v(x=0) = v_0
\end{align*}
\]

\[
\frac{v \, dv}{dx} = -\mu g - \frac{\gamma}{m} v = -(\mu g + \frac{\gamma}{m} v)
\]

\[
\int_{v_0}^{0} \frac{v \, dv}{\mu g + \frac{\gamma}{m} v} = \int_{0}^{d} -d \, dx
\]

\[
\Rightarrow d = \int_{0}^{v_0} \frac{\gamma}{m} \, dv
\]

\[
d = \int \frac{\frac{\gamma}{m} v_0 + mg}{\frac{\gamma}{m} v_0 + mg} \left( \frac{m}{\gamma} \, du \right)
\]

\[
\begin{align*}
& = \left( \frac{m}{\gamma} \right)^2 \int \left[ 1 - \frac{mg}{\nu} \right] \, du \\
& \quad = \left( \frac{m}{\gamma} \right)^2 \left[ \frac{\nu}{m} v_0 + mg - \frac{mg (\frac{\nu}{m} v_0 + mg)}{\nu} \right] \quad \text{(Chain Rule)}
\end{align*}
\]