\[
\frac{dy}{dt} = y'(t) = \text{slope of the tangent line}
\]
Problem 1. Identify the differential equation that corresponds to the given direction field. Based on the direction field, determine the behavior of $y$ as $t \to \infty$. If this behavior depends on the initial value of $y$ at $t = 0$, describe the dependency.

a. $y' = 2y - 3$  b. $y' = 2 + y$  c. $y' = y - 3$  d. $y' = y(y + 3)$  e. $y' = y(y - 3)$  f. $y' = 1 + 2y$  

$y = 2$ is the equilibrium solution: $y' = 0$ at $y = 2$

$y = -1$  $y' = -2 - y$  $y = -1 < 0$

$y = 3$  $y' = 2 - y$  $y = 3 - 3 < 0$

$y' = 0$  $y > 0$

$y = 0$  $y' = 0$

$y < 0$  $y' < 0$

$y' \to \infty$ if $y(0) > 2$

$y = 2$ if $y(0) = 2$

$y' \to -\infty$ if $y(0) < 2$

$y \to \infty$ for $y(0) > -2$

$y = -1$ for $y(0) = -2$

$y' \to -\infty$ for $y(0) < -2$

$y' \to 0$ if $y(0) < 0$

$y = 0$ if $y(0) = 0$

$y < 0$ if $y(0) > 0$

$y > 0$ if $y(0) < 0$
Problem 2. For small, slowly falling objects, the assumption that the drag force is proportional to the velocity is a good one. For larger, more rapidly falling objects, it is more accurate to assume that the drag force is proportional to the square of the velocity.

a. Write a differential equation for the velocity of a falling object of mass \( m \) if the magnitude of the drag force is proportional to the square of the velocity and its direction is opposite to that of the velocity.

b. Determine the limiting velocity after a long time.

c. If \( m = 10 \text{ kg} \), find the drag coefficient so that the limiting velocity is 49 m/s.

d. If \( v(0) = 0 \), find an expression for \( v(t) \) at any time.

e. Find the distance \( x(t) \) that the object falls in time \( t \).

f. Find the time \( T \) it takes the object to fall 300 m.

\[ m \cdot \frac{dv}{dt} = mg - \gamma v^2 \]  

\[ \text{Newton's Law} \]

\[ (\text{mass})(\text{acceleration}) = (\text{gravity}) - (\text{drag}) \]

\[ m \cdot \frac{dv}{dt} = mg - \gamma v^2 \]

A. \[ \text{drag} = \gamma v^2 \]

\[ m \cdot \frac{dv}{dt} = mg - \gamma v^2 \]

B. Solving \( \frac{dv}{dt} = 0 \) \( \Rightarrow \) \( 0 = mg - \gamma v^2 \) \( \Rightarrow \) \( v = \sqrt{\frac{mg}{\gamma}} \)

C. \( 0 = 10 \cdot 9.8 - \gamma \cdot (49)^2 \) \( \Rightarrow \) \( \gamma = \frac{9.8}{49^2} = \frac{2}{49} \)
Problem 3. A radioactive material, such as the isotope thorium-234, disintegrates at a rate proportional to the amount currently present. If \( Q(t) \) is the amount present at time \( t \), then \( \frac{dQ}{dt} = -rQ \), where \( r > 0 \) is the decay rate.

a. If 100 mg of thorium-234 decays to 82.04 mg in 1 week, determine the decay rate \( r \).

b. Find an expression for the amount of thorium-234 present at any time \( t \).

c. Find the time required for the thorium-234 to decay to one-half its original amount.

\[
\begin{align*}
\text{a. } & \quad \frac{dQ}{dt} = -rQ \\
\text{divide by } Q & \quad \frac{dQ}{Q} = -r \, dt \\
\text{integrate} & \quad \int \frac{dQ}{Q} = \int -r \, dt \\
\Rightarrow & \quad \ln(Q) = -rt + C_0 \\
\text{b. } & \quad Q(t) = Ce^{-rt} \\
\text{c. } & \quad T \text{ be the time.}
\end{align*}
\]
\[ Q(0) = Q_o = 100 \]
\[ Q(T) = \frac{1}{2} Q_o = 50 \quad \Leftrightarrow \quad 100 e^{-rT} = 50 \]
\[ \Leftrightarrow \quad e^{-rT} = \frac{1}{2} \]
\[ \Leftrightarrow \quad -rT = \ln(\frac{1}{2}) \]
\[ \Leftrightarrow \quad T = -\frac{\ln(\frac{1}{2})}{r} \quad r = -\frac{\ln(0.8204)}{7} \]
Problem 5. Determine the order of the given differential equation: also state whether the equation is linear or nonlinear.

1. \( \frac{d^2 y}{dx^2} + \frac{dy}{dx} + 2y = \sin x \)
   \( \text{order} = 2, \text{ linear} \)

2. \( \left( 1 + \frac{x^2}{y^3} \right) \frac{dy}{dx} + \frac{dy}{dx} \right|_{y = e^t} \)
   \( \text{order} = 2, \text{ nonlinear} \)

3. \( \frac{d^2 y}{dx^2} + \frac{dy}{dx} + 2y = 1 \)
   \( \text{order} = 4, \text{ linear} \)

4. \( \frac{d^2 y}{dt^2} + \sin(t + y) - \sin t \)
   \( \text{order} = 2, \text{ nonlinear} \)

5. \( 2y_{xyp} + 4y_{xyp} - 4y_{xyp} = 0 \)
   \( \text{order} = 4, \text{ linear} \)

6. \( \frac{dy}{dx} + 2y = 2 \frac{d^2 y}{dx^2} \)
   \( \text{order} = 2, \text{ nonlinear} \)
Problem 6. Verify that each given function is a solution of the differential equation.

\[ t^2 y'' + 5ty' + 3y = 0 \quad t > 0; \quad y_1(t) = t^2, \quad y_2(t) = t^{-2} \ln t \]

\[ y_2 = t^2 \ln t \]

\[ y_2' = (2t^{-3}) \ln t + (t^{-2}) \cdot \frac{1}{t} \]

\[ = -2t^{-3} \ln t + t^{-3} \]

\[ = t^{-3} (-2 \ln t + 1) \]

\[ y_2'' = (-3t^{-4})(-2 \ln t + 1) + (t^{-3}) (-2 \cdot \frac{1}{t}) \]

\[ = -3t^{-4} (-2 \ln t + 1) - 2t^{-4} \]

\[ = t^{-4} (6 \ln t - 5) \]

Plug \( y_2, y_2', y_2'' \) in the equation: \( t^2 y'' + 5ty' + 4y = 0 \)

\[ (t^2)(t^{-4})(6 \ln t - 5) + (5t)(t^{-3})(-2 \ln t + 1) + 4t^{-2} \ln t \]

\[ = t^{-2}(6 \ln t - 5) + 5t^{-2}(-2 \ln t + 1) + 4t^{-2} \ln t \]

\[ = 0 \Rightarrow y_2 \text{ is a solution}. \]

\( y_1: \text{skip}. \)
Problem 7. Determine the values of \( r \) for which the given differential equation has solutions of the form \( y = e^{rt} \).

\[
y'' + y' - 6y = 0
\]

\[
y = e^{rt} \Rightarrow y' = re^{rt}, \quad y'' = r^2e^{rt}
\]

Plugging in,

\[
(r^2e^{rt}) + (re^{rt}) - 6e^{rt} = 0
\]

\[
\Leftrightarrow (r^2 + r - 6)e^{rt} = 0
\]

\[
\Leftrightarrow r^2 + r - 6 = 0 \quad \text{ divide by } e^{rt} \quad \text{since } e^{rt} > 0 \text{ always.}
\]

\[
\Leftrightarrow (r+3)(r-2) = 0 \quad \Rightarrow \boxed{r = -3, 2} \quad \Rightarrow y = e^{3t}, e^{2t}
\]

Problem 8. Determine the values of \( r \) for which the given differential equation has solutions of the form \( y = t^r \) for \( t > 0 \).

\[
t^2y'' + 4ty' + 2y = 0.
\]

\[
y = t^r \]

\[
y' = rt^{r-1}
\]

\[
y'' = r(r-1)t^{r-2}
\]

\[
\Leftrightarrow t^2 \left( r(r-1) + 4r + 2r^2 \right) + 2t^r = 0
\]

\[
\Leftrightarrow r(r+1)r + 4rt + 2t^r = 0
\]

\[
\Leftrightarrow \boxed{(r^2 + 3r + 2)} t^r = 0
\]

\[
\Leftrightarrow r^2 + 3r + 2 = 0
\]

\[
\Rightarrow (r+2)(r+1) = 0
\]

\[
\Rightarrow \boxed{r = -2, -1} \quad \Rightarrow y = t^2, t^{-1}
\]
Problem 4. According to Newton's law of cooling, the temperature \( u(t) \) of an object satisfies the differential equation

\[
\frac{du}{dt} = -k(u - T),
\]

where \( T \) is the constant ambient temperature and \( k \) is a positive constant. Suppose that the initial temperature of the object is \( u(0) - u_0 \).

a. Find the temperature of the object at any time.

\[
\Rightarrow \frac{du}{u - T} = -k \, dt
\]

\[
\Rightarrow \int \frac{du}{u - T} = \int -k \, dt
\]

\[
\Rightarrow \ln |u - T| = -kt + C_0
\]

\[
\Rightarrow |u - T| = e^{-kt + C_0} = e^{-kt} \cdot C_1
\]

\[
\Rightarrow u - T = \pm C_1 e^{-kt} = C_2 e^{-kt}
\]

\[
\Rightarrow u = T + C_2 e^{-kt}
\]

General solution

\[
U(0) = u_0:
\]

\[
U_0 = T + C_2 \cdot e^{-k0}
\]

\[
\Rightarrow U_0 = T + C_2
\]

\[
\Rightarrow C_2 = U_0 - T
\]

Particular solution

\[
U(t) = T + (u_0 - T)e^{-kt}
\]
6. \( U(t) = \frac{1}{2} T + \frac{1}{2} U_0 \):

\[
\left( \frac{1}{2} T + \frac{1}{2} U_0 \right) = T + (U_0 - T) e^{-kT}
\]

\[\iff \frac{1}{2} U_0 - \frac{1}{2} T = (U_0 - T) e^{-kT}\]

\[\iff \frac{1}{2} (U_0 - T) = (U_0 - T) e^{-kT}\]

\[\iff \frac{1}{2} = e^{-kT}\]

\[\iff \ln\left(\frac{1}{2}\right) = -kT\]

\[\iff k = -\frac{\ln\left(\frac{1}{2}\right)}{T} = \frac{\ln(2)}{T}\]