Week in Review #10

1. A 2-kg mass is attached to a spring with the Hooke’s constant of 3 N/m, and it is subject to moving in a medium with a damping constant of 5 N-s/m. The mass is initially displaced +0.5 m from its equilibrium position and is released. Then after 2 seconds, a hammer hits the mass in such a way that its velocity suddenly changes by 5m/s in the positive direction (i.e., $\Delta v = +5 \text{ m/s}$).

   a) Find the impulse of the hammer and its units.
   b) Write the hammer force using an appropriate Dirac delta function. Note that your force must result in the same impulse value you found in part (a).
   c) Set up an IVP and find the position $x(t)$ of the mass for $t \geq 0$. 
2. Find the solution of the initial value problem
   \[(a) \ y'' - y = 4\delta(t - 2) + t^2, \quad y(0) = 0, \quad y'(0) = 2.\]
3. Find the following convolutions using the definition only.
   (a) \( e^t * e^{3t} \)

   (b) \( t * t^n \), where \( n = 0, 1, 2, \ldots \)

4. Using the Laplace transform (instead of the definition), compute the following convolutions.
   (a) \( u_a(t) * u_b(t) \)

   (b) \( t^n * t^m \), \( n, m = 0, 1, 2, \ldots \)
5. In each of the following cases find a function (or a generalized function) \( g(t) \) that satisfies the equality for \( t \geq 0 \).
   (a) \( t * g(t) = t^4 \)

   (b) \( 1 * 1 * g(t) = t^2 \)

   (c) \( 1 * g(t) = 1 \)

6. Write the inverse Laplace transform of the function \( F(s) = \frac{s}{(s+1)^2(s+4)^3} \) in terms of a convolution integral.
7. Solve the initial value problem \( y'' - 2y' - 3y = g(t), \ y(0) = 1, \ y'(0) = -3. \)
8. Determine the radius of convergence for the following power series:

(a) \[ \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \]

(b) \[ \sum_{n=1}^{\infty} \frac{(-1)^n n^2 (x + 2)^n}{3^n} \]
9. For the equation \((x^2 + 1)y'' + xy' - y = 0\)
   (a) Determine a lower bound for the radius of convergence of the series solutions of the differential equation about \(x_0 = 0\).
   (b) Seek its power series solution about \(x_0 = 0\); find the recurrence relation.
   (c) Find the general term of each solution \(y_1(x)\) and \(y_2(x)\).
   (d) Find the first four terms in each of two solutions \(y_1\) and \(y_2\). Show that \(W[y_1, y_2](0) \neq 0\).
10. For the following equation, determine $\phi''(x_0)$ and $\phi'''(x_0)$, for the given point $x_0$ if $y = \phi(x)$ is a solution of the given initial-value problem.

$$y'' + x^2 y' + (\sin x)y = 0; \quad y(0) = a_0, \quad y'(0) = a_1$$