1. Given the following differential equations and their corresponding direction field, determine the behavior as $t$ increases.

Fig. 1: $y'(t) = 2y(t) - 3$

$y(t) \to \infty$, if $y(0) > \frac{3}{2}$

$y(t) \to -\infty$, if $y(0) < \frac{3}{2}$

$y(t) = \frac{3}{2}$, if $y(0) = \frac{3}{2}$

Fig. 2: $y'(t) = y(t)(2 - y(t))$

$y(0) > 2 \Rightarrow y(t) \to 2$

$y(0) = 2 \Rightarrow y(t) = 2$

$0 < y(0) < 2 \Rightarrow y(t) \to 2$

$y(0) = 0 \Rightarrow y(t) = 0$

$y(0) < 0 \Rightarrow y(t) \to -\infty$

Find the equation of the linear solution of the last differential equation.

Let $y(t) = at + b$ be a solution of $y' = t - 1 - y$.

$a = t - 1 - (at + b)$

$\rightarrow a = t - 1 - a + b$

$a - 1 = 0 \Rightarrow a = 1$

$a + b + 1 = 0 \Rightarrow b = -2$

$y = t - 2$
2. Given the differential equation \( \frac{dy}{dt} = ty - 1 \).

(a) What is the slope of the graph of the solutions at \((0, 1)\), at the point \((1, 1)\), at the point \((3, -1)\), at the point \((0, 0)\)?

\[
m_1 = f(0, 1) = 0(1) - 1 = -1
\]

\[
m_2 = f(1, 1) = 1(1) - 1 = 0
\]

\[
m_3 = f(3, -1) = 3(-1) - 1 = -4
\]

\[
m_4 = f(0, 0) = 0(0) - 1 = -1
\]

(b) Find all the points where the tangents to the solution curves are horizontal.

\[
y' = 0 \implies ty - 1 = 0 \quad \text{on the curve} \quad y = \frac{1}{t}
\]

\[
\{ (t, \frac{1}{t}) \mid t \neq 0 \}
\]

(c) Describe the nature of the critical points.

\[
y'' = \frac{d}{dt} \left( ty - 1 \right) = y' + ty'' = \frac{1}{t}
\]

At the critical point, \( y'' = 0 \).

If \( y > 0 \) \( \implies y'' = y > 0 \) \( \implies \) local minimum

If \( y < 0 \) \( \implies y'' = y < 0 \) \( \implies \) local maximum

3. The instantaneous rate of change of the temperature \( T \) of coffee at time \( t \) is proportional to the difference between the temperature \( M \) of the air and the temperature \( T \) at time \( t \).

(a) Find the mathematical model for the problem.

\[
\frac{dT}{dt} = k(M - T) = -k(T - M)
\]

Units of \( k \): \( \frac{1}{\text{sec}} \)
(b) Given that the room temperature is 75° and $k = 0.08$, find the solutions to the differential equation.

$$M = 75, \quad K = 0.08$$

$$\int \frac{dT}{75 - T} = \int 0.08 \, dt$$

$$\ln |75 - T| = 0.08t + K$$

$$\ln |75 - T| = -0.08t - K \Rightarrow 175 - T = e^{-0.08t - K} = e^{-k} e^{-0.08t}$$

$$75 - T = \pm e^{-k} e^{-0.08t}$$

$$T = 75 - (\pm e^{-k} e^{-0.08t})$$

(c) The initial temperature of the coffee is 200°F. Find the solution to the problem.

$$T(0) = 200 = 75 + C e^{-0.08(0)} \Rightarrow C = 200 - 75 = 125$$

$$\Rightarrow T(t) = 75 + 125 e^{-0.08t}$$

$$\lim_{t \to \infty} T(t) = 75$$

4. Your swimming pool containing 60,000 gal of water has been contaminated by 5 kg of a non toxic dye that leaves a swimmer’s skin an unattractive green. The pool’s filtering system can take water from the pool, remove the dye, and return the water to the pool at a flow rate of 200 gal/min.

(a) Write down the initial value problem for the filtering process; let $q(t)$ be the amount of dye in the pool at any time $t$.

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$$\left\{ \begin{array}{l}
\frac{dq}{dt} = -\frac{200}{60,000} q(t) \\
q(0) = 5 \quad \text{kg}
\end{array} \right.$$
(c) You have invited several dozen friends to a pool party that is scheduled to begin in 4 hours. You have also determined that the effect of the dye is imperceptible if its concentration is less than 0.02 g/gal. Is your filtering system capable of reducing the dye concentration to this level within 4 hours?

\[ q_{\text{min}}(240) = 5 e^{-\frac{240}{360}} \approx 1.72 \text{ kg} \]

\[ \frac{1.72}{60,000} = 2.87 \times 10^{-5} \frac{\text{kg}}{\text{gal}} = 0.0287 \frac{\text{g}}{\text{gal}} > 0.02 \frac{\text{g}}{\text{gal}} \]

4 hours not enough!

5. The direction field for the differential equation

\[ x'(t) = \frac{2tx(t)}{1 + x(t)} \quad \frac{dx}{dt} = f(t, x) \] normal

is given below.

Sketch the graph of the solutions to the initial value problems

(a) \( x(0) = 1 \)
(b) \( x(0) = -2 \)
(c) \( x(0) = -0.5 \)
6. Match the direction field to the differential equations

\[ a) \quad y' = \frac{y - 2}{y} \quad b) \quad y' = 2 - y \quad c) \quad y' = 2 + y \]
\[ d) \quad y' = -2 - y \quad e) \quad y' = (y - 2)^2 \quad f) \quad y' = (y + 2)^2 \]
7. Given the following differential equations, classify each as an ordinary differential equation, partial differential equation, give the order. If the equation is an ordinary differential equation, say whether the equation is linear or non linear.

(a) \( \frac{dy}{dx} = 3y + x^2 \)

(b) \( 5 \frac{d^4y}{dx^4} + y = x(x - 1) \)

(c) \( \frac{\partial N}{\partial t} = \frac{\partial^2 N}{\partial r^2} + \frac{1}{r} \frac{\partial N}{\partial r} + kN \)

(d) \( \frac{dx}{dt} = x^2 - t \)

(e) \( (1 + y^2) y'' + ty' + y = e^t \)

8. (a) Show that \( f(x) = (x^2 + Ax + B)e^{-x} \) is solution to \( y'' + 2y' + y = 2e^{-x} \) for all real numbers \( A \) and \( B \).

\[
\begin{align*}
    f' &= (2x+A)e^{-x} + (x^2+A+B)(-e^{-x}) = (2x+A)e^{-x} - (x^2+A+B)e^{-x} \\
    f'' &= 2e^{-x} + (2x+A)(-e^{-x}) - (2x+A)e^{-x} + (x^2+A+B)e^{-x} \\
    f'' + 2f' + f &= \underbrace{2e^{-x} - 2(2x+A)e^{-x} + (x^2+A+B)e^{-x}}_{\text{term}} + \underbrace{2(2x+A)e^{-x} - 2(x^2+A+B)e^{-x}}_{\text{term}} + \underbrace{(x^2+A+B)e^{-x}}_{\text{term}} = 2e^{-x}.
\end{align*}
\]

(b) Find a solution that satisfies the initial condition \( y(0) = 3 \) and \( y'(0) = 1 \).

\[
\begin{align*}
    f(0) = 3 &\Rightarrow (0^2 + A \cdot 0 + B)e^0 = 3 \Rightarrow B = 3 \\
    f'(0) = 1 &\Rightarrow A e^0 - Be^0 = 1 \Rightarrow A = 3 + 1 = 4
\end{align*}
\]

\[ f(x) = y = (x^2 + 4x + 3)e^{-x} \]
9. Determine for which values of $r$ the function $t^r$ is a solution of the differential equation

$$t^2 y'' - 4ty' + 4y = 0, \quad t > 0.$$ 

Let $y = t^r$, then $y' = rt^{r-1}$, and $y'' = r(r-1)t^{r-2}$.

$$t^2 \cdot r(r-1)t^{r-2} - 4t \cdot rt^{r-1} + 4t^r = 0, \quad t > 0$$

$$[r(r-1) - 4r + 4] t^r = 0 \quad \Rightarrow \quad r^2 - 5r + 4 = 0$$

$$(r-1)(r-4)=0$$

$$\Rightarrow \begin{cases} y_1 = t \\ y_2 = t^4 \end{cases}$$

10. For which values of $r$ is the function $(x-1)e^{-rx}$ solution to $y'' - 6y' + 9y = 0$?

Let $y = (x-1)e^{-rx}$.

$$y' = e^{-rx} + (x-1) (-re^{-rx})$$

$$y'' = -re^{-rx} - re^{-rx} + (x-1) r^2 e^{-rx}$$

$$y'' - 6y' + 9y = 0 \Rightarrow -2re^{-rx} + (x-1) r^2 e^{-rx} - 6e^{-rx} + 6r(x-1)e^{-rx} + q(x-1)e^{-rx} = 0$$

$$\begin{bmatrix} -2r + r^2(x-1) - 6 + 6r(x-1) + q(x-1) \end{bmatrix} = 0 \quad \text{for all } x$$

$$x(r^2 + 6r + q) - (r^2 + 8r + 15) = 0 \quad \text{for all } x$$

$$(r+3)^2 = 0 \quad \Rightarrow \quad r = -3$$

$$r^2 + 8r + 15$$

$$= (-3)^2 - 8(-3) + 15$$

$$= 0$$
11. In curling, the player has to slide a stone of mass $m$ on a smooth surface with friction coefficient $\mu$. If the air friction is proportional to the velocity of the stone with a drag coefficient of $\gamma$, find the stopping time and the stopping distance of the stone if the initial velocity is $v_0$.

$$F = ma$$

$$\begin{cases} 
- \mu mg - \gamma v = m \frac{dv}{dt} \\
\quad v(0) = v_0
\end{cases}$$

$$v = v(t)$$

$$T = \text{stopping time}$$

$$v(T) = 0$$

$$\frac{dv}{dt} = -\mu g - \frac{\gamma}{m} v \implies \int \frac{dv}{\mu g + \frac{\gamma}{m} v} = \int -dt$$

$$\frac{m}{\gamma} \ln \left( \mu g + \frac{\gamma}{m} v \right) = -t + K \implies \ln \left( \mu g + \frac{\gamma}{m} v \right) = -\frac{\gamma}{m} t + \frac{\gamma K}{m}$$

$$\mu g + \frac{\gamma}{m} v = e^{-\frac{\gamma}{m} t}$$

$$\frac{\gamma}{m} v = e^{-\frac{\gamma}{m} t} - \mu g \implies v = \left( \frac{m}{\gamma} e^{-\frac{\gamma t}{m}} \right) e^{-\frac{\gamma}{m} t} - \frac{\mu mg}{\gamma}$$

$$v(t) = C e^{-\frac{\gamma}{m} t} - \frac{\mu mg}{\gamma}$$

$$v(0) = v_0$$

$$v_0 = C - \frac{\mu mg}{\gamma} \implies C = v_0 + \frac{\mu mg}{\gamma}$$

$$v = \left( v_0 + \frac{\mu mg}{\gamma} \right) e^{-\frac{\gamma}{m} t} - \frac{\mu mg}{\gamma}$$

$$0 = \left( v_0 + \frac{\mu mg}{\gamma} \right) e^{-\frac{\gamma}{m} T} - \frac{\mu mg}{\gamma} \implies e^{-\frac{\gamma}{m} T} = -\frac{\mu mg/\gamma}{v_0 + \mu mg/\gamma}$$
\[
\frac{x}{m} T = \ln\left(\frac{v_0 + \frac{\mu mg}{Y}}{\mu mg}\right) = \ln\left(\frac{v_0 Y}{\mu mg} + 1\right)
\]

\[
T = \frac{m}{\gamma} \ln\left(\frac{v_0 Y}{\mu mg} + 1\right)
\]

\[
\frac{dx}{dt} = v = \left(v_0 + \frac{\mu mg}{Y}\right)e^{-\frac{Y}{m} t} - \frac{\mu mg}{Y}
\]

\begin{align*}
\varphi & \quad \{ \\
\chi(0) &= 0
\end{align*}

\Rightarrow \chi(t) = \int v \, dt = \left(v_0 + \frac{\mu mg}{Y}\right) \left(-\frac{m}{Y}\right) e^{-\frac{Y}{m} t} + \frac{Y}{m} t + K

\chi(0) = 0

\Rightarrow 0 = \left(v_0 + \frac{\mu mg}{Y}\right) \left(-\frac{m}{Y}\right) (0) + K

\Rightarrow K = \frac{m}{Y} \left(v_0 + \frac{\mu mg}{Y}\right)

**Stopping distance** \(= \chi(T)\)

\[
= \frac{m}{Y} \left(v_0 + \frac{\mu mg}{Y}\right) - \frac{\mu mg}{Y} \cdot \frac{m}{Y} \ln\left(\frac{v_0 Y}{\mu mg} + 1\right)
\]

\[
- \frac{m}{Y} \left(v_0 + \frac{\mu mg}{Y}\right) e^{-\frac{Y}{m} T} \ln\left(\frac{v_0 Y}{\mu mg} + 1\right)
\]

\[
= \left(\frac{m}{Y}\right)^2 \left[ \frac{Y}{m} v_0 + \mu g - \mu g \left(\frac{Y}{m} v_0 + \mu g\right) - \mu g + \mu g \ln(\mu g) \right]
\]