Section 4.3: Rules of Probability

- The probability of an event, \( A \), is always between 0 and 1 inclusively. \( 0 \leq P(A) \leq 1 \)
- The probability of an impossible event is 0. \( P(\emptyset) = 0 \)
- The probability of the certain event is 1. \( P(S) = 1 \)
- Union Rule: \( P(A \cup B) = P(A) + P(B) - P(A \cap B) \)
- Complement Rule: \( P(A^C) = 1 - P(A) \)

Pr 1. Let \( S = \{s_1, s_2, s_3, s_4\} \) be the sample space for an experiment with the distribution given below.

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_1 )</td>
<td>( \frac{3}{50} )</td>
</tr>
<tr>
<td>( s_2 )</td>
<td>( \frac{4}{25} )</td>
</tr>
<tr>
<td>( s_3 )</td>
<td>( \frac{12}{50} )</td>
</tr>
<tr>
<td>( s_4 )</td>
<td>( \frac{5}{25} )</td>
</tr>
</tbody>
</table>

Let \( A = \{s_1, s_3\} \) and \( B = \{s_1, s_4\} \).

(a) Fill in the missing probability in the distribution table.

\[
P(s_2) = 1 - \left( \frac{3}{50} + \frac{4}{25} + \frac{12}{50} \right) = 1 - \frac{29}{50} = \frac{21}{50}
\]

Determine the following probabilities.

(b) \( P(A) = P(s_1) + P(s_3) \)
\[
= \frac{3}{50} + \frac{12}{50} = \frac{15}{50} = \frac{3}{10}
\]

(c) \( P(A \cup B) = P(s_1) + P(s_3) + P(s_4) \)
\[
= \frac{3}{50} + \frac{12}{50} + \frac{5}{25} = \frac{20}{50} = \frac{4}{10} = 1 - P(s_2)
\]

(d) \( P(A \cap B) = P(s_1) \)
\[
= \frac{3}{50}
\]

(e) \( P((A \cap B)^C) = 1 - P((A \cap B)) \)
\[
= 1 - \frac{3}{50} = \frac{47}{50}
\]

(f) \( P(A^C \cup B) = P(s_1) + P(s_2) + P(s_4) \) or \( 1 - P(s_3) \)
\[
= \frac{3}{50} + \frac{4}{25} + \frac{5}{25} = \frac{29}{50}
\]
Pr 2. A fair standard five-sided die is rolled, noting the number shown, and a fair coin was flipped, noting the side showing. What is the probability that
(a) An even number is rolled or a tails was showing.
\[ P(\text{Even} \cup \text{Tails}) = \frac{7}{10} \]
(b) A 3 is rolled and a tails is showing.
\[ P(3 \cap \text{tails}) = \frac{1}{10} \]
(c) The coin does not show a heads.
\[ P(\bar{H}) = \frac{5}{10} \]

Pr 3. Let \( A \) and \( B \) be two events of an experiment. Suppose \( P(A) = 0.65, P(B) = 0.62, \) and \( P(A \cup B) = 0.84. \)
Calculate the following probabilities:
(a) \( P(A^c) \)
\[ A^c = \{y, z\} \]
\[ 1 - P(A) = 1 - 0.65 = 0.35 \]
(b) \( P(A \cap B) \)
\[ P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.65 + 0.62 - 0.84 = 0.43 \]
(c) \( P(A^c \cup B^c) \)
\[ A^c \cup B^c = \{x, y, z\} \]
\[ = \{x, y, z\} \]
\[ = 0.22 + 0.19 + 0.16 = 0.57 \]
(d) \( P((A \cup B)^c) \)
\[ = 1 - P(A \cup B) \]
\[ = 1 - 0.84 \]
\[ = 0.16 \]
\[ P((A \cup B)^c) = P(\bar{A} \cap \bar{B}) \]
\[ P(\bar{A} \cap \bar{B}) = 0.16 \]
Section 4.4: Probability Distributions

- Random Variables
- Expected Value $E(X) = x_1p_1 + x_2p_2 + \cdots x_np_n$
- Premiums
- Fair Games $E(\text{net winnings}) = 0$

Pr 1. The probability distribution for tossing a coin three times and counting the number of heads is given below.

<table>
<thead>
<tr>
<th>$X$</th>
<th>$P(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{1}{8}$</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{3}{8}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{3}{8}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{8}$</td>
</tr>
</tbody>
</table>

(a) Compute the probability that more than one head is tossed.

$$P(X > 1) = P(2) + P(3)$$
$$= \frac{3}{8} + \frac{1}{8}$$
$$= \frac{4}{8}$$

(b) Compute the probability that four heads are tossed.

$$P(X = 4) = 0$$

(c) State the expected number of heads in an experiment where three coins are tossed.

$$E(X) = 0(\frac{1}{8}) + 1(\frac{3}{8}) + 2(\frac{3}{8}) + 3(\frac{1}{8})$$
$$= 0 + \frac{3}{8} + \frac{6}{8} + \frac{3}{8}$$
$$= \frac{12}{8}$$
$$= \frac{3}{2}$$
Pr 2. You are going on a European vacation and decide to purchase travel insurance on your brand new luggage worth $1500. The insurance policy will cost $48. In the event your luggage is damaged to the point of needing duct tape, then you will receive 50% of the value of the luggage. In the event your luggage is lost or stolen, then you will receive 100% of the value of the luggage. According to airline data, the probability of your luggage being damaged and needing duct tape is 1%, while the probability your luggage is lost or stolen is 0.8%. Let $X$ be the insurance company’s net gain or loss on the policy described.

(a) Create a probability distribution for $X$.

<table>
<thead>
<tr>
<th>X</th>
<th>Needs duct tape</th>
<th>lost/stolen</th>
<th>nothing happens</th>
</tr>
</thead>
<tbody>
<tr>
<td>$48 - \frac{750}{1500} = 30$</td>
<td>$48 - 1500 = -1452$</td>
<td>$48 - 0 = 48$</td>
<td></td>
</tr>
<tr>
<td>$0.01$</td>
<td>$0.008$</td>
<td>$1 - (0.01 + 0.008) = 0.982$</td>
<td></td>
</tr>
</tbody>
</table>

(b) Compute the insurance company’s expected profit for this policy.

$$E(X) = -30(0.01) + (-1452)(0.008) + 48(0.982)$$

$$= 28.50$$

Expected net gain of $28.50$

Pr 3. You purchase a brand new yacht for $1,000,000 and insure it. The policy pays 40% of the yacht’s value if it is involved in a minor accident (as defined by the insurance company) or 90% of the yacht’s value if it sinks. The probability of a minor yachting accident is 0.25, while the probability your yacht sinks is 0.005. What is the minimum amount the insurance company will charge for this policy.

(a) Create a probability distribution for $X$.

<table>
<thead>
<tr>
<th>X</th>
<th>minor</th>
<th>sinks</th>
<th>nothing happens</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p - 400000$</td>
<td>$p - 900000$</td>
<td>$p - d$</td>
<td></td>
</tr>
<tr>
<td>$0.25$</td>
<td>$0.005$</td>
<td>$1 - (0.25 + 0.005) = 0.745$</td>
<td></td>
</tr>
</tbody>
</table>

$$E(X) = 0.25 (p - 400000) + 0.005 (p - 900000) + 0.745 (p)$$

$$= 0.25p - 100000 + 0.005p - 4500 + 0.745p$$

$$= 0.745p - 104500$$

$$p = 104,500$$

(b) Premium is $\$104,500$. 

"premium - payout"
Pr 4. A real estate investor buys a parcel of land for $225,000. He estimates the probability that he can sell it for $375,000 to be 24%, the probability that he can sell it for $200,000 to be 35%, and the probability that he can sell it for $189,000 to be 41%. What is the expected profit from the sale of this land?

\[
\begin{array}{c|c|c|c|c}
\text{Profit} & 375000 & -225000 & 200000 & -225000 \\
\text{P(X)} & 0.24 & 0.35 & 0.41 & \\
E(X) = 0.24(375000) + 0.35(-225000) + 0.41(-360000) = \$12,490
\end{array}
\]

Pr 5. You play a game where a card is drawn from a well-shuffled standard deck of 52 cards, noting the color of the card, and a spinner divided into four equal regions (red, blue, green, and yellow) is spun, noting the color. If the spinner lands on a color other than yellow, you win $3. If the color of the card is red and the spinner lands on red, you win $10. Otherwise you lose. Let \( X \) be your winnings.

a. Create a probability distribution for \( X \).

<table>
<thead>
<tr>
<th>( X )</th>
<th>( \text{not yellow} )</th>
<th>( \text{red} )</th>
<th>( \text{lose} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(X)</td>
<td>( \frac{6}{8} )</td>
<td>( \frac{1}{8} )</td>
<td>( \frac{1}{8} )</td>
</tr>
</tbody>
</table>

b. Compute your expected winnings for the game.

\[
E(X) = 3 \left( \frac{6}{8} \right) + 10 \left( \frac{1}{8} \right) + 0 \left( \frac{1}{8} \right) \\
= \frac{18}{8} + \frac{10}{8} + 0 \\
= \frac{28}{8} \\
= 3.50
\]

c. How much should be charged in order to make the game fair?

\[
E(\text{not winning}) = 0
\]

Net winning: Win - paid to play

\( \times \) You should pay $3.50 for the game to be fair.