Section 2.2

Section 2.2 gives a method to solve differential equations can be written in the form:

$$f(y)\frac{dy}{dx} = g(x).$$

Since y is a function of x, the LHS looks like the derivative $\frac{d}{dx}F(y)$ of some function F. This is just the chain rule:

$$\frac{d}{dx}F(y) = F'(y)\frac{dy}{dx}.$$

So, $f(y) = F'(y)$, or, equivalently, $F(y) = \int f(y)dy$. So the first step in solving this equation is to find the anti–derivative of f. Then the LHS can be recognized as:

$$\frac{d}{dx}F(y) = g(x).$$

And so we have:

$$F(y) = \int g(x)dx + C,$$

and it is possible that we can solve this equation for y in terms of x.

Section 2.2

Problem 1. Find the general solution to $y' = x^2/y$.

First, get the y stuff on one side and the x stuff on the other:

$$y\frac{dy}{dx} = x^2.$$

Recall that y is a function of x and so the LHS looks like $\frac{d}{dx}F(y)$ by the chain rule. The function $F(y) = \int y dy = \frac{1}{2}y^2$ and so the equation is the same as:

$$\frac{d}{dx}(\frac{1}{2}y^2) = x^2,$$

taking anti–derivatives of both sides gives:

$$\frac{1}{2}y^2 = \frac{1}{3}x^3 + C.$$

There are a number of ways to write this. For example:

$$3y^2 - 2x^3 = C.$$

Observe that for the ODE to be valid, we must have $y \neq 0$.

Section 2.2

Problem 2. Find the general solution to $\frac{dy}{dx} = \frac{x-e^{-x}}{y+e^{y}}$.

This is:

$$(y + e^y)\frac{dy}{dx} = x - e^{-x}.$$

We recognize the LHS as:

$$\frac{d}{dx}\left(\int y + e^y dy\right) = \frac{d}{dx}\left(\frac{1}{2}y^2 + e^y\right),$$

and so the ODE can be written as:

$$\frac{d}{dx}\left(\frac{1}{2}y^2 + e^y\right) = x - e^{-x},$$

integrating both sides:

$$\frac{1}{2}y^2 + e^y = \frac{1}{2}x^2 + e^{-x} + C.$$

This can be written as:

$$y^2 - x^2 + 2(e^y - e^{-x}) = C.$$

Observe that for the ODE to be valid, we must have $e^y + y \neq 0$.

Section 2.2

Problem 3. Solve the IVP:

$$y' = \frac{1-2x}{y}, \qquad y(1) = -2.$$

Plot the general solution and determine the interval on which the solution is defined.

Re–write it as:

$$y\frac{dy}{dx} = 1 - 2x.$$

We recognize the LHS as $\frac{d}{dx}\int y dy = \frac{d}{dx}\frac{1}{2}y^2$ and so the ODE is:

$$\frac{d}{dx}\frac{1}{2}y^2 = 1 - 2x,$$

integrating both sides gives:

$$\frac{1}{2}y^2 = x - x^2 + C,$$

which can be written as:

$$y^2 + 2x^2 - 2x = C.$$

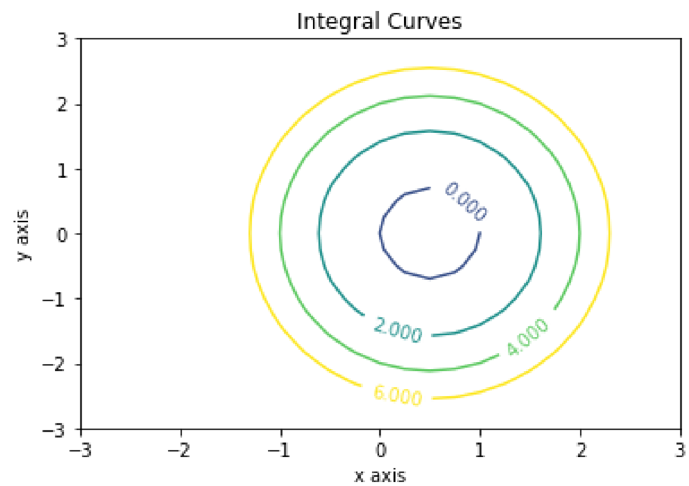
Finding for C:

$$(-2)^2 + 2(1)^2 - 2(1) = 4.$$

So the solution is given implicitly by:

$$y^2 + 2x^2 - 2x = 4.$$

Section 2.2



The plot to the left plots the curves. We are interested in the one with the label 4.000 which corresponds to $C = 4$. Since our condition is $y(1) = -2$, we want the bottom part of this ellipse. For that part of the curve:

$$y = -\sqrt{-2x^2 + 2x + 4}.$$

This is going to be valid as long as the expression in the square root is non-negative. That is, for $-1 < 2 < x$.

This interval can be determined in a different way as well. Notice that expressions like:

$$y^2 + 2x^2 - 2x = 4$$

define a curve in the (x, y) plane. The portions of the curve that pass the vertical line test define y implicitly as a function of x. E.g. the bottom part of the circle defines y implicitly as a function of x, but the right part of a circle doesn't because it fails the vertical line test.

A portion of the curve fails the VLT whenever the tangent line is vertical at a point on the portion of the curve. So, for example, the right hand side of the circle $x^2 + y^2 = 1$ has a vertical tangent at $(1, 0)$. However, if we look at the bottom part of the curve (that is where $y < 0$), there is no vertical tangent line. And so the curve doesn't fail the VLT.

From calc 3, for a curve defined by:

$$F(x, y) = C$$

the gradient $\nabla F(x, y) = \langle F_x, F_y \rangle$ is perpendicular to the curves and so the curve has a vertical tangent whenever ∇F is horizontal and this happens when $F_y = 0$. Now, from the equation $f(y)\frac{dy}{dx} = g(x)$, the function $f = F_y$. So, there is a vertical tangent whenever $f(y) = 0$. Then, using C, we can determine x.

In this problem $f(y) = y$ and so there are vertical tangent lines whenever $y = 0$. So, for this

$$0 + 2x^2 - 2x = 4.$$

Notice this is the exact expression we analyzed above and has solutions $x = -1, 2$. This means the curve has vertical tangents at the points $(-1, 0)$ and $(2, 0)$.

Section 2.2

Problem 4. Solve the IVP:

$$\frac{dy}{dx} = \frac{2x}{y + x^2 y}, \qquad y(0) = -2.$$

Plot the general solution and determine the interval on which the solution is defined.

First, write this as:

$$y\frac{dy}{dx} = \frac{2x}{1+x^2}.$$

The LHS we recognize as $\frac{d}{dx}(\frac{1}{2}y^2)$ so the ODE is:

$$\frac{d}{dx}\frac{1}{2}y^2 = \frac{2x}{1+x^2},$$

integrating both sides gives:

$$\frac{1}{2}y^2 = \ln(1+x^2) + C.$$

And this can be written as:

$$y^2 - 2\ln(1+x^2) = C.$$

To compute C:

$$(-2)^2 - 2\ln(1+0^2) = 4.$$

So this is:

$$y^2 - 2\ln(1+x^2) = 4.$$

We can solve this for y:

$$y = \pm\sqrt{4 + 2\ln(1+x^2)}.$$

Since $y(0) = -2$, we want the negative solution:

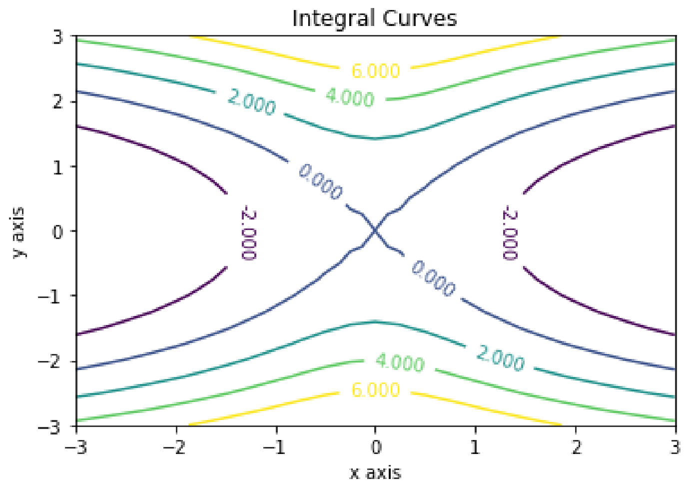
$$y = -\sqrt{4 + 2\ln(1+x^2)}.$$

We know that there is a vertical tangent line when $0 = f(y) = y$. Solving for x:

$$0^2 - 2\ln(1+x^2) = 4 \Longrightarrow \ln(1+x^2) = -2 \Longrightarrow 1 + x^2 = e^{-2} \Longrightarrow x = \pm\sqrt{e^{-2} - 1}.$$

These are imaginary numbers and so there are no vertical tangent lines.

Section 2.2



We care about the one labeled with 4.000. Notice that – at least from this picture – it looks like for this curve the tangent line is never vertical.

Section 2.2

Problem 5. Solve the IVP:

$$(\sin 2x)dx + (\cos 3y)dy = 0 \qquad y(\frac{\pi}{2}) = \frac{\pi}{3}.$$

Plot the general solution and determine the interval on which the solution is defined.

This is the same as:

$$\cos(3y)\frac{dy}{dx} = -\sin 2x.$$

Integrating both sides:

$$\int \cos(3y)dy = -\int \sin 2x dx$$

gives:

$$\frac{1}{3}\sin(3y) = \frac{1}{2}\cos(2x) + C.$$

Which can be written as:

$$2\sin(3y) - 3\cos(2x) = C.$$

Finding C:

$$2\sin(\pi) - 3\cos(\pi) = 3.$$

Thus:

$$2\sin(3y) - 3\cos(2x) = 3.$$

Solving:

$$\sin(3y) = \frac{3}{2}(1 + \cos(2x)) \Longrightarrow y = \frac{1}{3}\arcsin(\frac{3}{2}(1 + \cos(2x))) = \frac{1}{3}\arcsin(3\cos^2 x).$$

This isn't exactly correct because we need to make sure the argument to arcsin is between -1 and 1 and we need to specify which branch of arcsin to take.

Section 2.2

We need to have:

$$3\cos^2 x \leq 1,$$

which is the same as:

$$\cos^2 x < \frac{1}{3}.$$

And this is:

$$-\frac{1}{\sqrt{3}} \leq \cos x \leq \frac{1}{\sqrt{3}}$$

This means that x must be in the interval $[\arccos\frac{1}{\sqrt{3}}, \arccos\frac{-1}{\sqrt{3}}]$.

Now we need to pick which "branch" of arcsin to take. By convention, arcsin has a range between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (the "right side" of the unit circle). Since $y(\frac{\pi}{2}) = \frac{\pi}{3}$, and since $y = \frac{1}{3}\arcsin(3\cos^x)$, we need to have arcsin have range $\frac{\pi}{2}$ to $\frac{3\pi}{2}$ (the "left side" of the unit circle) and so we add π to it. (Note that the range of any inverse sin function needs to be either the right side or the left side of the circle, or it would fail the vertical line test).

Thus, the final answer is:

$$y = \frac{1}{3}(\pi + \arcsin(3\cos^x)), \qquad x \in [\arccos\frac{1}{\sqrt{3}}, \arccos\frac{-1}{\sqrt{3}}].$$

Section 2.3

Problem 6. Consider a tank used in an experiment. After one experiment, there is 200L of a dye solution with concentration of 1 gram per liter. The tank is rinsed with fresh water flowing in at 2 liters per minute and water is flowing out of the tank at the same rate. How long until the concentration is 1% of the original?

Let $D(t)$ be the grams of dye in the tank at time t. We want to solve $D(t) = (.01)(200)(1) = 2$ for t. First, we need to find D. It is modeled by the ODE:

$$\frac{dD}{dt} = \text{rate in} - \text{rate out} = (0)(2) - \frac{D(t)}{200}2 = -\frac{1}{100}D(t).$$

So the relevant IVP is:

$$\frac{dD}{dt} = -\frac{1}{100}D, \qquad\qquad D(0) = 200.$$

The general solution is $D(t) = Ae^{-\frac{1}{100}t}$. Since there is 200 grams of dye to begin with, the solution to the IVP is $D(t) = 200e^{\frac{-1}{100}t}$.

Now we solve:

$$2 = 200e^{\frac{-1}{100}t} \implies \frac{1}{100} = e^{\frac{-1}{100}t} \implies -\ln 100 = -\frac{1}{100}t \implies t = 100\ln 100.$$

Section 2.3

Problem 7. A tank has 100 gallons of water and 50 oz of salt. Water containing salt with a concentration of $\frac{1}{4}(1 + \frac{1}{2}\sin t)$ ounces / gallon flows in at a rate of 2 gal / min and the mixture flows out at the same rate. Find the amount of salt at time t. What is $\lim S(t)$?

We want to find $S(t)$ and the ODE that models it is:

$$\frac{dS}{dt} = \text{rate in} - \text{rate out} = \frac{1}{4}(1 + \frac{1}{2}\sin t)2 - \frac{S(t)}{100}2.$$

So the relevant IVP is:

$$\frac{dS}{dt} + \frac{1}{50}S(t) = \frac{1}{2}(1 + \frac{1}{2}\sin t), \qquad S(0) = 50.$$

Solve the ODE using integrating factors. It is $\mu(t) = e^{\frac{1}{50}t}$, multiplying by this gives:

$$(Se^{\frac{1}{50}t})' = e^{\frac{1}{50}t}\frac{1}{2}(1 + \frac{1}{2}\sin t).$$

Integrating both sides:

$$S(t)e^{\frac{1}{50}t} = 25e^{\frac{1}{50}t} - \frac{1}{4}\frac{50e^{\frac{t}{50}}(50\cos t - \sin t)}{2501} + C,$$

whence:

$$S(t) = 25 + \frac{25}{5002}(\sin t - 50\cos t) + Ce^{-\frac{1}{50}t}$$