Section 2.4

Section 2.4 is about uniqueness and existence theorems. The first one is about first order linear equations:

Theorem 1. *If the functions p and g are continuous on an open interval $I = (\alpha, \beta)$ containing the point $t = t_0$, then there exists a unique function $x = \varphi(t)$ that solves the Initial Value Problem:*

$$x'(t) + p(t)x(t) = g(t), \qquad\qquad x(t_0) = x_0.$$

The next one is about more general first order equations.

Theorem 2. *Let the functions f and $\frac{\partial f}{\partial x}$ be continuous in some rectangle $\alpha < t < \beta$ and $\gamma < x < \delta$ that contains the point (t_0, x_0). Then in some interval $t_0 - h < t < t_0 + h$, contained in (α, β) there is a unique solution $x = \varphi(t)$ to the IVP:*

$$x' = f(t, x) \qquad x(t_0) = x_0.$$

Section 2.4

Problem 1. Determine – without solving the problem – an interval in which the solution of the given IVP is certain to exist:

$$(t-3)y' + (\ln t)y = 2t, \qquad y(1) = 2.$$

First we need to write this in the form of the theorem:

$$y' + \frac{\ln t}{t-3}y = \frac{2t}{t-3}, \qquad y(1) = 2.$$

To use the first theorem above, note that:

$$p(t) = \frac{\ln t}{t-3}, \qquad g(t) = \frac{2t}{t-3}.$$

The function $p(t)$ is continuous everywhere except at 0 and 3. The function $g(t)$ is continuous everywhere except 3. So, the functions are continuous on $(-\infty, 0)$, $(0, 3)$ and $(3, \infty)$. The $t_0 = 1$ so a solution is sure to exist on $(0, 3)$.

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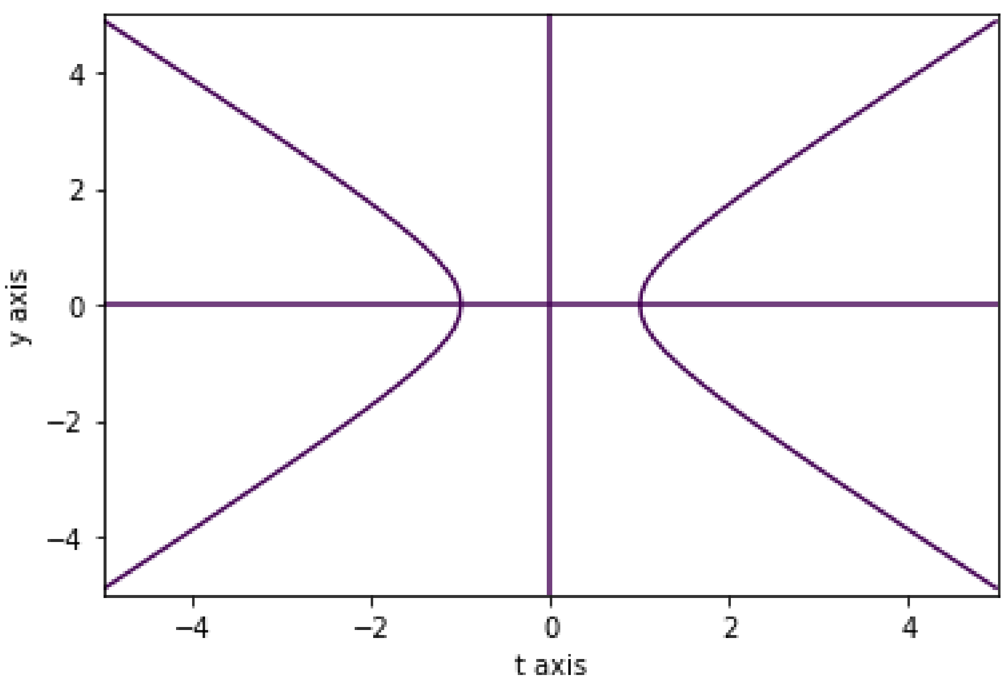
Problem 2. Determine the region in the ty plane in which the hypotheses are the second theorem above are satisfied.

$$y' = \frac{\ln|ty|}{1-t^2+y^2}.$$

The numerator is not continuous when $t=0$ and $y=0$. And the denominator is not continuous when $1-t^2+y^2=0$. The partial derivative with respect to y is:

$$\frac{-t^2-2y^2\ln(ty)+y^2+1}{y(1-t^2+y^2)^2}.$$

This continuous in the same places as the function above.

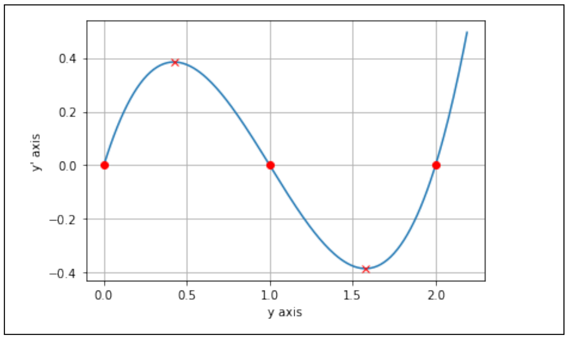


Section 2.5

Section 2.5

Problem 3. Sketch several solution curves for the ODE:

$$y' = y(y-1)(y-2), y_0 \geq 0.$$



Note that:

$$f'(y) = 3y^2 - 6y + 2.$$

Setting this to zero we get:

$$y_{1,2} = 1 \pm \frac{1}{\sqrt{3}}.$$

x Interval	$x' = f(x)$	$x'' = f'(x)f(x)$	Inc/Dec	CU/CD
$(0, 1-\frac{1}{\sqrt{3}})$	$+$	$+$	Inc	CU
$(1-\frac{1}{\sqrt{3}}, 1)$	$+$	$-$	Inc	CD
$(1, 1+\frac{1}{\sqrt{3}})$	$-$	$+$	Dec	CU
$(1+\frac{1}{\sqrt{3}}, 2)$	$-$	$-$	Dec	CD
$(2, \infty)$	$+$	$+$	Inc	CU

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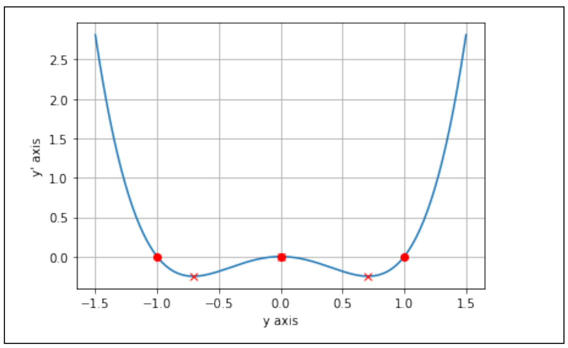
Problem 4. Sketch several solution curves for the ODE:

$$y' = y^2(y^2 - 1), -\infty < y_0 < \infty.$$



Note that:

$$f'(y) = 3y^2 - 6y + 2.$$

Setting this to zero we get:

$$\pm\frac{1}{\sqrt{2}}, 0$$

x Interval	$x' = f(x)$	$x'' = f'(x)f(x)$	Inc/Dec	CU/CD
$(-\infty, -1)$	$+$	$-$	Inc	CD
$(1, -\frac{1}{\sqrt{2}})$	$-$	$+$	Dec	CU
$(-\frac{1}{\sqrt{2}}, 0)$	$-$	$-$	Dec	CD
$(0, \frac{1}{\sqrt{2}})$	$-$	$+$	Dec	CU
$(\frac{1}{\sqrt{2}}, 1)$	$-$	$-$	Dec	CD
$(1, \infty)$	$+$	$+$	Inc	CU

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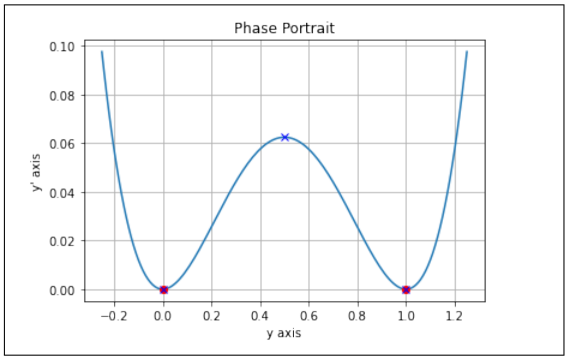
Problem 5. Sketch several solution curves for the ODE:

$$y' = y^2(1-y)^2, -\infty < y_0 < \infty.$$



Note that:

$$f'(y) = y(y-1)(2-4y)$$

Setting this to zero we get:

$$0, \frac{1}{2}, 1.$$

x Interval	$x' = f(x)$	$x'' = f'(x)f(x)$	Inc/Dec	CU/CD
$(-\infty, 0)$	$+$	$-$	Inc	CD
$(0, \frac{1}{2})$	$+$	$+$	Inc	CU
$(\frac{1}{2}, 1)$	$+$	$-$	Inc	CD
$(1, \infty)$	$+$	$+$	Inc	CU

Section 2.5

Section 2.6

If $F = F(x, y)$ and $y = y(x)$, then the multi variable chain rule says:

$$\frac{d}{dx}F(x, y) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}.$$

Section 2.6

We want to study equations of the form:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0.$$

The idea is that we want $M = F_x$ and $N = F_y$ for some function $F = F(x,y)$. If so, then this equation can be written:

$$\frac{d}{dx}F(x,y) = 0,$$

whence a solution is:

$$F(x,y) = C,$$

and this defines y *implicitly* as a function of x.

So, how can we determine that $M = F_x$ and $N = F_y$. Recall that $F_{xy} = F_{yx}$, so if $M = F_x$ and $N = F_y$ then it is true that $M_y = N_x$ and, indeed, if M and N satisfy this equation, then they satisfy $M = F_x$ and $N = F_y$. In this case, we say the equation is *exact*.

Section 2.6

Determine if the equation is exact and find the solution:

$$(3x^2 - 2xy + 2) + (6y^2 - x^2 + 3)y' = 0.$$

$$M_y(x, y) = -2x = N_x(x, y).$$

So this is exact. To find $F(x, y)$:

$$F(x, y) = \int (3x^2 - 2xy + 2)dx + h(y) = x^3 - x^2y + 2x + h(y).$$

To find h:

$$6y^2 - x^2 + 3 = -x^2 + h'(y) \Longrightarrow h'(y) = 6y^2 + 3 \Longrightarrow h(y) = 2y^3 + 3y + C.$$

So the solution is:

$$x^3 - x^2y + 2x + 2y^3 + 3y = C$$

Section 2.6

Section 2.6

Determine if the equation is exact and find the solution:

$$(ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + 2x) + (xe^{xy}\cos(2x) - 3)y' = 0.$$

$$M_y(x,y) = e^{xy}\cos(2x) + xye^{xy}\cos(2x) - 2xe^{xy}\sin(2x)$$
$$N_x(x,y) = e^{xy}\cos(2x) + yxe^{xy}\cos(2x) - 2xe^{xy}\sin(2x),$$

since these are equal, the equation is exact. Then:

$$F(x,y) = \int (xe^{xy}\cos(2x) - 3)dy + h(x) = e^{xy}\cos(2x) - 3y + h(x).$$

To find $h(x)$:

$$ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + 2x = ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + h'(x)$$

whence:

$$h'(x) = 2x \implies h(x) = x^2 + C.$$

So:

$$e^{xy}\cos(2x) + x^2 - 3y = C.$$

Section 2.6

Section 2.6

Find the solution (explicitly) to the IVP and the domain on which the solution is valid.

$$(2x - y) + (2y - x)y' = 0, \qquad y(1) = 3.$$

This is exact. To find F:

$$F(x, y) = \int (2x - y)dx + h(y) = x^2 - xy + h(y),$$

so:

$$2y - x = -x + h'(y) \Longrightarrow h'(y) = 2y \Longrightarrow h(y) = y^2 + C.$$

So:

$$x^2 - xy + y^2 = C.$$

To find C:

$$C = 1^2 - (1)(3) + 3^2 = 7.$$

So:

$$x^2 - xy + y^2 = 7.$$

Solving for y:

$$\begin{aligned} y^2 - xy = 7 - x^2 &\Longrightarrow (y - \frac{x}{2})^2 - \frac{x^2}{4} = 7 - x^2 \\ &\Longrightarrow (y - \frac{x}{2})^2 = 7 - \frac{3}{4}x^2 = \frac{28 - 3x^2}{4} \\ &\Longrightarrow y = \frac{x}{2} \pm \frac{1}{2}\sqrt{28 - 3x^2}. \end{aligned}$$

The initial condition gives:

$$3 = \frac{1}{2} \pm \frac{1}{2}\sqrt{28 - 3} = \frac{1}{2} \pm \frac{5}{2}.$$

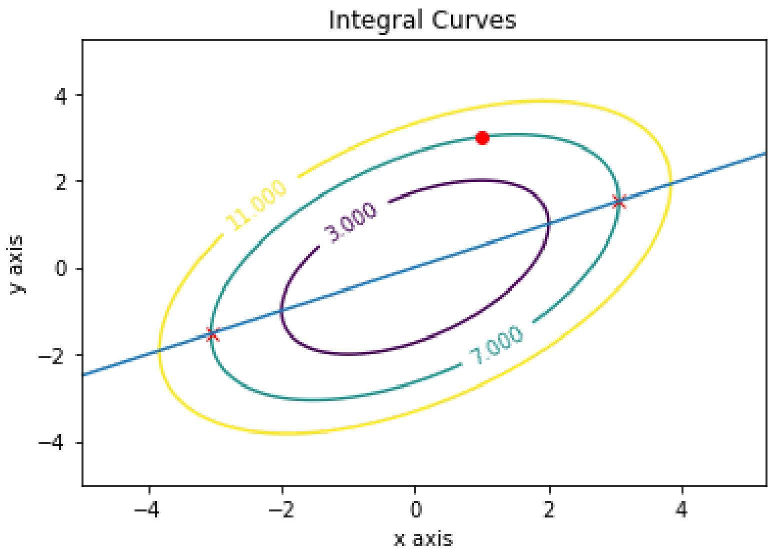
It is the "+" that is true so:

$$y = \frac{x}{2} + \frac{1}{2}\sqrt{28 - 3x^2}.$$

For the domain of validity, we need:

$$28 - 3x^2 \geq 0 \Longrightarrow x^2 < \frac{28}{3} \Longrightarrow |x| < \sqrt{\frac{28}{3}} \simeq 3.06$$

Section 2.6



The blue elipse is the one that corresponds to $y(1) = 3$ (this is the red dot). The red x's on the curve indicate the endpoints of the domain of validity. So the top blue curve is the curve that corresponds to the IVP. The line graphs the points where the tangent is vertical.

Section 2.6

Find an integrating factor to solve:

$$(3x^2y + 2xy + y^3) + (x^2 + y^2)y' = 0.$$

This is not exact. But, can we multiply this by a function $\mu(x)$ that will make it exact? What would such an equation need to satisfy:

$$\begin{aligned}\frac{\partial}{\partial y}(\mu(y)M(x,y)) = \frac{\partial}{\partial x}(\mu(y)N(x,y)) &\Longrightarrow \mu(x)M_y(x,y) = \mu'(x)N(x,y) + \mu(x)N_x(x,y)\\ &\Longrightarrow \mu'(x)N(x,y) = \mu(x)(M_y(x,y) - N_x(x,y))\\ &\Longrightarrow \frac{\mu'(x)}{\mu(x)} = \frac{M_y(x,y) - N_x(x,y)}{N(x,y)}.\end{aligned}$$

The only way this last line makes sense is if $(M_y - N_x)/N$ is a function of x alone. If not, then this makes no sense and there isn't an IF that is a function of x alone (in which case this formula isn't valid since we used the fact that $\mu_y = 0$).

For our particular problem:

$$\frac{M_y(x,y) - N_x(x,y)}{N(x,y)} = \frac{(3x^2 + 2x + 3y^2) - (2x)}{x^2 + y^2} = 3.$$

Thus:

$$\frac{\mu'(x)}{\mu(x)} = 3,$$

whence $\mu(x) = e^{3x}$. Multiplying by both sides gets:

$$e^{3x}(3x^2y + 2xy + y^3) + e^{3x}(x^2 + y^2) = 0,$$

and this is exact:

$$F(x,y) = \int e^{3x}(x^2 + y^2)dy + h(x) = e^{3x}(x^2y + \frac{1}{3}y^3) + h(x).$$

To find h:

$$e^{3x}(3x^2y + 2xy + y^3) = e^{3x}(2xy) + 3e^{3x}(x^2y + \frac{1}{3}y^3) + h'(x)$$

so:

$$h'(x) = 0 \Longrightarrow h(x) = C.$$

So the solution is:

$$e^{3x}(3x^2y + y^3) = C.$$

Section 2.6