Improper Integrals

Recall that an integral over an unbounded region is called *improper* and is defined using limits of ordinary integrals:

\[ \int_{t=a}^{\infty} f(t) dt := \lim_{b \to \infty} \int_{t=a}^{b} f(t) dt, \]

when this limit exists. When it doesn’t exist, we say that the improper integral diverges.

**Example.** Find \( \int_{t=0}^{\infty} e^{ct} dt \), if it exists.

It is defined as:

\[ \int_{t=0}^{\infty} e^{ct} dt = \lim_{b \to \infty} \int_{t=0}^{b} e^{ct} dt = \lim_{b \to \infty} \frac{1}{c} (e^{cb} - 1) = \infty. \]

Since this diverges, it does not exist. Since the limit is \( \infty \), we can also say that the integral is equal to \( \infty \).
Example. Find $\int_{t=0}^{\infty} e^{-ct} dt$, if it exists.

\[
\int_{t=0}^{\infty} e^{-ct} dt = \lim_{b \to \infty} \int_{t=0}^{b} e^{-ct} dt = \lim_{b \to \infty} \frac{1}{c} (1 - e^{-ct}) = \frac{1}{c}.
\]

Example. Find $\int_{t=0}^{\infty} e^{-ct} dt$, if it exists.

We use integration by parts below:

\[
\int_{t=0}^{\infty} te^{-t} dt = \lim_{b \to \infty} \left[ \int_{t=0}^{b} te^{-t} dt \right] = \lim_{b \to \infty} \left( -te^{-t} \big|_{t=0}^{t=b} + \int_{t=0}^{t=b} e^{-t} dt \right) = \lim_{b \to \infty} \left( -\frac{b}{e^b} + 1 \right) = 1.
\]
Taylor Series

Recall that a power series centered at $x_0$ is a series of the form:

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n.$$  

Here, the coefficients $a_n$ are constant (i.e. not functions of $x$, but they do change with $n$. For example:

$$\sum_{n=0}^{\infty} n(x - 1)^n$$  

is a power series centered at 1 and $a_n = n$.

A power series is said to converge at a point $x$ if

$$\lim_{M \to \infty} \sum_{n=0}^{M} a_n (x - x_0)^n$$

converges for that $x$.

A power series is said to converge absolutely at a point $x$ if

$$\lim_{M \to \infty} \sum_{n=0}^{M} |a_n (x - x_0)^n|$$

converges for that $x$.

Let

$$\lim_{n \to \infty} \left| \frac{a_{n+1}(x - x_0)^{n+1}}{a_n(x - x_0)^n} \right| = |x - x_0| \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = |x - x_0| L.$$  

If $|x - x_0| L < 1$ the series converges absolutely. If $|x - x_0| L > 1$, the series diverges. If $|x - x_0| L = 1$, this test gives no information.
Taylor Series
Taylor Series
Taylor Series

For which values of $x$ does the power series:

$$\sum_{n=1}^{\infty} (-1)^{n+1} n(x - 2)^n$$

converge? Using the test discussed above, we compute:

$$\lim_{n \to \infty} \frac{|x - 2| |n + 1|}{|n|} = |x - 2|.$$ 

This converges absolutely for $|x - 2| < 1$ and diverges for $|x - 1| > 1$. When $|x - 1| = 1$, this test gives no information. Remember in Calc 2, you had to test the endpoints to see what the interval of convergence is. But for now, it is enough to know that this series converges on $(1, 3)$ and that the ROC is 1.
Taylor Series

If a series converges on an interval, then its term-by-term differentiated series converges on that interval and is equal to the derivative of the series. That is, if

\[ f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n \]

converges on \((x_0 - R, x_0 + R)\) then:

\[ f'(x) = \sum_{n=0}^{\infty} na_n(x - x_0)^{n-1} = \sum_{n=1}^{\infty} na_n(x - x_0)^{n-1}. \]

And so this means the term-by-term twice differentiated series converges on the same interval and is equal to \(f''\) there:

\[ f''(x) = \sum_{n=1}^{\infty} n(n - 1)(x - x_0)^{n-2} = \sum_{n=2}^{\infty} n(n - 1)(x - x_0)^{n-2}. \]
Taylor Series

Recall that if:

\[ f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n \]

then

\[ a_n = \frac{f^{(n)}(x_0)}{n!} \].

Find the taylor series for \( f(x) = e^x \) centered at 0. Using the above formula, \( a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!} \). So it is:

\[ e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n. \]

In general, if \( f(x) \) is equal to its Taylor series on \( (x_0 - R, x_0 + R) \), then:

\[ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n. \]
Taylor Series
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