Chapter 1

Problem 1. Solve the IVP $\frac{dx}{dt} = -x + 5$, $x(0) = x_0$.

We can re–write this as:

$$\frac{\frac{dx}{dt}}{5 - x} = 1.$$

By the chain rule, the LHS is the derivative of $-\ln|5 - x|$. So the equation becomes:

$$-\frac{d}{dt}\ln|5 - x| = 1,$$

and integrating both sides with respect to t gives:

$$\ln|5 - x| = -t + C.$$

Raising both sides to the power of e and letting $A = e^C$:

$$x - 5 = \pm Ae^{-t}.$$

$\pm$ is really an arbitrary constant that can be absorbed into the A. Adding 5, we get:

$$x(t) = Ae^{-t} + 5.$$

To find the value of A, use the initial condition $x(0) = x_0$:

$$x_0 = A + 5,$$

so $A = x_0 - 5$ and the final solution is:

$$x(t) = (x_0 - 5)e^{-t} + 5.$$

Chapter 1

Problem 2. Solve the IVP $\frac{dx}{dt} = -2x + 10$, $x(0) = x_0$.

$$x(t) = 5 + (x_0 - 5)e^{-2t}.$$

Chapter 1

Problem 3. Determine the values of r for which the given differential equation has solutions of the form $x = e^{rt}$. The equation is:

$$x' + 2x = 0$$

Plugging $x(t) = e^{rt}$ into the equation we get:

$$re^{rt} + 2e^{rt} = 0.$$

This is the same as:

$$r + 2 = 0,$$

whence $r = -2$. So, the only values of r for which e^{rt} could possibly be a solution are $r = -2$. Now, notice that if $x = e^{-2t}$ then:

$$x'(t) + 2x = -2e^{-2t} + 2e^{-2t} = 0,$$

which verifies that $x(t) = e^{-2t}$ is a solution. So the answer is $r = -2$.

Chapter 1

Problem 4. Do the same question as above, except with the equation:

$$x''' - 3x'' + 2x' = 0.$$

Plugging $x = e^{rt}$ into this equation gives:

$$r^3 e^{rt} - 3r^2 e^{rt} + 2re^{rt} = 0.$$

Dividing by e^{rt}, this gives:

$$0 = r^3 - 3r^2 + 2r = r(r^2 - 3r + 2).$$

We know that $r = 0, 1, 2$ are solutions to this. So the only values of r for which e^{rt} could be a solution are $0, 1, 2$. And, indeed, the functions e^{rt} for $r = 0, 1, 2$ are solutions. So the answer is $r = 0, 1, 2$.

Section 2.1

Problem 5. Find the general solution to $x' + 3x = t + e^{-2t}$.

We will use integrating factors. First, find the integrating factor. Remember, this is the function $\mu(t)$ that satisfies $\mu x' + \mu 3x = (\mu x)'$. Expanding this equation gives:

$$\mu x' + \mu 3x = \mu' x + \mu x', \longrightarrow \mu 3x = \mu' x \longrightarrow \frac{\mu'}{\mu} = 3 \longrightarrow \mu(t) = e^{3t}.$$

Multiplying the original equation by the integrating factor gives:

$$(e^{3t}x(t))' = te^{3t} + e^t.$$

Integrating both sides gives:

$$e^{3t}x(t) = \frac{1}{3}e^{3t}(t - \frac{1}{3}) + e^t + C,$$

whence:

$$x(t) = \frac{1}{3}(t - \frac{1}{3}) + e^{-2t} + Ce^{-3t}.$$

Notice this involves an integration by parts. For the integral:

$$\int te^{3t}dt,$$

let $u = t$, $\frac{dv}{dt} = e^{3t}$ and by the IBP formula, this is:

$$\int te^{3t}dt = \frac{t}{3}e^{3t} - \frac{1}{3}\int e^{3t}dt = \frac{t}{3}e^{3t} - \frac{1}{9}e^{3t} = \frac{1}{3}e^{3t}(t - \frac{1}{3}).$$

Section 2.1

Problem 6. Find the general solution to $x' + x = te^{-t} + 1$.

$$x(t) = Ce^{-t} + 1 + \frac{t^2}{2}e^{-t}.$$

Section 2.1

Problem 7. Solve the IVP $x' + \frac{2}{t}x = \frac{\cos t}{t^2}$, $x(\pi) = 0$, $t > 0$.

We will use integrating factors again. The integrating factor equation is:

$$(x\mu)' = x'\mu + x\mu' = \mu x' + \mu\frac{2}{t}x.$$

Subtracting $x'\mu$ from both sides and then dividing by x leads to:

$$\mu' = \mu\frac{2}{t}.$$

Which is:

$$\frac{\mu'}{\mu} = \frac{2}{t}.$$

The LHS is $\frac{d}{dt}\ln|\mu|$ and integrating both sides gives:

$$\ln|\mu| = 2\ln|t|.$$

And so $\mu(t) = t^2$. Multiplying both sides of the original equation by μ gives:

$$(t^2 x(t))' = \cos t.$$

Integrating both sides gives:

$$t^2 x(t) = \sin t + C \longrightarrow x(t) = \frac{\sin t}{t^2} + \frac{C}{t^2}.$$

To find C we use the initial condition:

$$0 = x(\pi) = \frac{C}{\pi^2},$$

whence $C = 0$ and so the solution is:

$$x(t) = \frac{\sin t}{t^2}.$$

Section 2.1

Problem 8. Solve the IVP $tx' + (t+1)x = t$, $x(\ln 2) = 1$, $t > 0$.

First, we divide both sides by t (this is possible since $t > 0$):

$$x' + \frac{t+1}{t}x = 1.$$

We will use integrating factors. Again the equation for the IF is:

$$(x(t)\mu(t))' = x'\mu + x\mu' = \mu x' + \mu\frac{t+1}{t}x,$$

which simplfies to:

$$\mu' = \mu\frac{t+1}{t}.$$

So this is just:

$$\frac{d}{dt}\ln|\mu| = \frac{t+1}{t}.$$

Integrating both sides we get:

$$\ln|\mu(t)| = t + \ln|t| + C \longrightarrow |\mu(t)| = e^{t+\ln|t|+C}.$$

Thus $\mu(t) = te^t$. Multiplying both sides by μ is:

$$(te^t x(t))' = te^t.$$

Integrating both sides gives:

$$te^t x(t) = \int te^t = te^t - \int e^t dt = te^t - e^t + C.$$

Thus:

$$x(t) = 1 - \frac{1}{t} + \frac{C}{te^t}.$$

To find C use the initial condition:

$$1 = 1 - \frac{1}{\ln 2} + \frac{C}{\ln 2 e^{\ln 2}}.$$

So $C = 2$ and:

$$x(t) = 1 - \frac{1}{t} + \frac{2}{te^t}.$$

Section 2.1

Problem 9. Describe the behavior of the solution corresponding to the initial value a_0 for the IVP $x' - \frac{1}{2}x = 2\cos t$, $x(0) = a_0$.

Using integrating factors, the IF equation is:

$$\frac{d}{dt}\ln|\mu(t)| = -\frac{1}{2} \longrightarrow \mu(t) = e^{-\frac{1}{2}t}.$$

Multiplying both sides of our DE gives:

$$(e^{-\frac{1}{2}t}x(t))' = 2e^{-\frac{1}{2}t}\cos t.$$

Integrating both sides gives:

$$e^{-\frac{1}{2}t}x(t) = -\frac{4}{5}e^{-\frac{1}{2}t}\cos t + \frac{8}{5}e^{-\frac{1}{2}t}\sin t + C.$$

Thus:

$$x(t) = -\frac{4}{5}\cos t + \frac{8}{5}\sin t + Ce^{\frac{1}{2}t}.$$

Finding C in terms of a_0:

$$a_0 = -\frac{4}{5} + C,$$

and so:

$$x(t) = -\frac{4}{5}\cos t + \frac{8}{5}\sin t + (a_0 + \frac{4}{5})e^{\frac{1}{2}t}.$$

If $a_0 + \frac{4}{5} > 0$ then the solutions go to infinity. If not, the solutions go to negative infinity. In the critical case $a_0 + \frac{4}{5} = 0$, solutions oscillate.