Section 2.3: Systems of Two Equations in Two Unknowns

1. State the type of linear system given without graphing or actually computing the solution. Then, state the number of solutions.

\[ \begin{align*}
L_1: & \quad 4x + 5y = 12 \\
L_2: & \quad 2x = -\frac{3}{2}y + 6
\end{align*} \]

\[ \begin{align*}
5y &= -4x + 12 \\
y &= -\frac{4}{5}x + \frac{12}{5}
\end{align*} \]

\[ m_1 = -\frac{4}{5}, \quad b_1 = \frac{12}{5} \]

\[ \begin{align*}
-\frac{5}{2}y &= 2x - 6 \\
y &= -\frac{4}{5}x + \frac{12}{5}
\end{align*} \]

\[ m_2 = -\frac{4}{5}, \quad b_2 = \frac{12}{5} \]

As \( m_1 = m_2 \) AND \( b_1 = b_2 \), the system is dependent and has infinitely many solutions.

2. Find the value of \( k \) so that the following system of equations has no solution.

\[ \begin{align*}
L_1: & \quad y = \frac{5}{4}x + 1 \\
L_2: & \quad 10x - ky = -6
\end{align*} \]

\[ \begin{align*}
L_1: & \quad y = \frac{5}{4}x + 1 \\
m_1 &= \frac{5}{4} \quad \text{but} \quad b_1 \neq b_2
\end{align*} \]

\[ \begin{align*}
L_2: & \quad 10x - ky = -6 \\
-ky &= -10x - 6 \\
y &= \frac{10}{k}x + \frac{6}{k}
\end{align*} \]

\[ m_2 = \frac{10}{k}, \quad b_2 = \frac{6}{k} \]

\[ m_1 = m_2 \]

\[ \frac{5}{4} = \frac{10}{k} \]

\[ 5k = 40 \]

\[ k = 8 \]

Check

\[ b_1 \neq b_2, \text{ when } k=8 \]

\[ \frac{5}{4} \neq \frac{6}{8} \]

\[ 1 \neq \frac{6}{8} \quad \text{True} \]

So when \( k=8 \) the system has no solution.
3. Solve each system using the stated method. Write any solutions as ordered pairs with exact values. For parametric solutions use $p$ as your parameter.

(a) \[
\begin{align*}
3x + 2y &= 5 \\
y &= -\frac{3}{2}x + 2
\end{align*}
\] using the graphical method.

\[
\begin{align*}
L_1: & \quad 3x + 2y = 5 \\
x\text{-int: } & \quad (\frac{5}{3}, 0) \\
y\text{-int: } & \quad (0, \frac{5}{2}) \\
L_2: & \quad y = -\frac{3}{2}x + 2 \\
x\text{-int: } & \quad (\frac{4}{3}, 0) \\
y\text{-int: } & \quad (0, 2)
\end{align*}
\]

No Solution

(b) \[
\begin{align*}
3x - 2y &= -3 \\
5x - y &= 2
\end{align*}
\] using the substitution method.

\[
\begin{align*}
3x - 2y &= -3 \\
5x - y &= 2 \\
\Rightarrow \quad & y = 5x - 2 \\
3x - 2(5x - 2) &= -3 \\
3x - 10x + 4 &= -3 \\
-7x &= -7 \\
x &= 1 \\
y &= 5(1) - 2 \\
y &= 3
\end{align*}
\]

\[(x, y) = (1, 3)\]

(c) \[
\begin{align*}
3x - 2y &= -4 \\
4y &= 6x + 8
\end{align*}
\] using the addition method.

\[
\begin{align*}
(3x - 2y &= -4) \cdot 2 \Rightarrow & 6x - 4y = -8 \\
4y &= 6x + 8 \\
\Rightarrow & -6x + 4y = 8
\end{align*}
\]

\[
0x + 0y = 0
\]

0 = 0 True \Rightarrow Infinitely many Solutions

Let \(y = p\), where \(p\) is any real number.

then

\[
\begin{align*}
3x - 2y &= -4 \\
3x &= 2p - 4 \\
x &= \frac{2}{3}p - \frac{4}{3}
\end{align*}
\]

\[
\begin{align*}
\text{So } (x, y) &= \left(\frac{2}{3}p - \frac{4}{3}, p\right)
\end{align*}
\]
4. The production cost for a record company are $18 per record and if they produce 60 records, then the total costs are $1652. The company sells each record for $40. Determine and interpret the break-even point for the record company on the production and sale of these records.

\[
\text{Cost: } C(x) = 18x + F \\
\text{and } (60, 1652) \\
1652 = 18(60) + F \\
572 = F \\
\text{Cost: } C(x) = 18x + 572 \\
\text{Revenue: } R(x) = 40x \\
50 \quad R(x) = C(x) \\
40x = 18x + 572 \\
22x = 572 \\
x = 26 \\
R(26) = 40(26) = 1040 \\
The company will cover their all their costs of $1040 when 26 records are sold. The company will earn a profit gain when 27 or more records are produced and sold. \\
\text{Break-even point: } (x, R) = (26, 1040)
\]

5. Consumers will buy 10,000 items at a price of $120 per item. If the price goes up by $30 per item, then they will only buy 7600 items. Producers will not market this item below $40, but if the price per item increases by $15, the producers will provide 6000 items to the market. Determine and interpret the market equilibrium point for these items.

\[
\text{Demand: } (10000, 120) \\
(7600, 120 + 30) \\
D(x) = \frac{150 - 120}{7600 - 10000} = \frac{30}{-2400} = -\frac{1}{80} \\
D(x) = -\frac{1}{80}(x - 10000) + 120 = \frac{1}{80}x + 245 \\
\text{Supply: } (0, 40) \\
(6000, 40 + 25) \\
S(x) = \frac{65 - 40}{6000 - 0} = \frac{25}{4000} = \frac{1}{240} \\
S(x) = \frac{1}{240}(x - 0) + 40 = \frac{1}{240}x + 40 \\
\text{Equilibrium } P+: \quad D(x) = S(x) \\
\frac{-\frac{1}{80}x + 245}{= \frac{1}{240}x + 40} \\
205 = \frac{1}{60}x \\
12300 = x \\
\text{The equilibrium point is } (12300, 91.25). \\
\text{If producers make 12,300 items and sell them at } $91.25 \text{ each,} \\
\text{consumers will purchase all 12,300 items.}
1. Set up, but do not solve, a system of linear equations which could be used to solve the problem.

You have $50,000 to invest in Fund A and Fund B. Fund A pays 7.4% and Fund B pays 9.8%. How much do you invest in each to get a return of $4,072 per year?

\[
\begin{align*}
\alpha & : = \text{the amount, in dollars, invested in Fund A} \\
\beta & : = \text{the amount, in dollars, invested in Fund B} \\
\alpha + \beta & = 50000 \quad \text{(Total invested)} \\
0.074\alpha + 0.098\beta & = 4072 \quad \text{(Returns)}
\end{align*}
\]

2. Write the corresponding augmented matrix for the systems of linear equations.

\[
\begin{bmatrix}
4x + 2y + 3z = 72 \\
2y - 3z = 12 \\
-x + 9 = 5y + z \Rightarrow +x + 5y + z = 9
\end{bmatrix}
\]

3. Write a system of linear equations which corresponds to the augmented matrix. Assume the variables are \( x \) and \( y \) or \( x \), \( y \), and \( z \).

\[
\begin{bmatrix}
-2 & -6 & -10 & -12 \\
0 & 1 & 2 & 3 \\
2 & 1 & 2 & -5
\end{bmatrix}
\]

\[
\begin{align*}
-2x - 6y - 10z & = -12 \\
y + 2z & = 3 \\
2x + y + 2z & = -5
\end{align*}
\]
4. Perform the indicated row operation and write the resulting matrix.
\[
\begin{bmatrix}
1 & -4 & 1 \\
5 & 2 & 19
\end{bmatrix}
-5R_1 + R_2 \rightarrow R_2
\]
\[
\begin{array}{c}
-5R_1 = -5 & 20 & -5 \\
+ R_2 = 5 & 2 & 19 \\
\hline
0 & 22 & 14
\end{array}
\]
\[
\begin{bmatrix}
1 & -4 & 1 \\
0 & 22 & 14
\end{bmatrix}
\]

5. State if the matrix is in reduced row-echelon form. If the matrix is not in reduced row-echelon form, state which of the four conditions is first violated, as stated in the definition.

(a) \[
\begin{bmatrix}
1 & 3 & -2 \\
0 & 1 & 0
\end{bmatrix}
\]
Not in rref
The column with a leading 1 should have 0's as all other entries in the column.

(b) \[
\begin{bmatrix}
1 & 2 & 0 & 3 \\
0 & 0 & 1 & -5 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
In rref
6. Solve each system of linear equations using matrices. Write your answer as an ordered pair or ordered triple, as appropriate. For parametric solutions use \( t \) as your parameter.

(a) \[
\begin{align*}
3x + 5y &= -2 \\
-9x - 15y &= 6
\end{align*}
\]

\[
\begin{bmatrix}
3 & 5 & | & -2 \\
-9 & -15 & | & 6
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 5/3 & | & -2/3 \\
0 & 0 & | & 0
\end{bmatrix} \Rightarrow \begin{cases}
x + \frac{5}{3}y = \frac{-2}{3} \\
0 = 0
\end{cases}
\]

Let \( y = t \), where \( t \) is any real number.

Then \( x + \frac{5}{3}y = \frac{-2}{3} \)

\( x + \frac{5}{3}t = \frac{-2}{3} \)

\( x = \frac{-5}{3}t - \frac{2}{3} \)

Solution: \( (x, y) = \left( \frac{-5}{3}t - \frac{2}{3}, t \right) \)

(b) \[
\begin{align*}
x + 4z &= 0 \\
x + y &= -2z + 1 \\
6z - 3x &= -3y + 15
\end{align*}
\]

\[
\begin{bmatrix}
1 & 0 & 4 & | & 0 \\
1 & 1 & 2 & | & 1 \\
-3 & 3 & 6 & | & 15
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 0 & | & -2 \\
0 & 1 & 0 & | & 2 \\
0 & 0 & 1 & | & \frac{1}{2}
\end{bmatrix}
\]

Solution: \( (x, y, z) = (-2, 2, \frac{1}{2}) \)