Note ♯6 (Sequences, Series)

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[Sequences]

(1) Determine whether each sequence converges or diverges. If it converges, find the limit.

(a) $a_n = \dfrac{11 - 2n^4}{3n^6 + 17n^2}$

(b) $b_n = \dfrac{7n + n^3}{17n - 9n^2}$

(c) $c_n = \ln(6n^5 + 13n) - \ln(8n^5 - 4)$

(d) $a_n = \sin\left(\dfrac{3\pi n + (-\frac{1}{3})^n}{18n}\right)$

(e) $b_n = \sqrt{5e^{9/n} + 4}$

(f) $c_n = \dfrac{(-1)^n 2n^8}{3n^8 + 9n^2 - 16}$

(g) $a_n = \dfrac{(-1)^{n-1}3n^2}{n^3 - 4n}$

(h) $c_n = \dfrac{\sin(5n)}{n^2 + 7}$

Monotonic Sequence Theorem

Every bounded, monotonic sequence is convergent.

(2) Determine whether the following sequences are increasing, decreasing, or not monotonic. Also, determine if the sequence is bounded.

(a) $a_n = \dfrac{5}{n^2 + 7}$

(b) $b_n = \dfrac{(-3)^n}{2^{2n}}$

(c) $a_n = \dfrac{3n}{\ln(4n)}$

(3) Assume that the sequence defined below is decreasing and $a_n \geq 0$ for all $n \geq 1$. Determine if the sequence is convergent or divergent and why. If the sequence is convergent, find its limit.

$$a_1 = 2, \qquad a_{n+1} = \frac{6}{7 - a_n}$$

[Series]

The geometric series

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots + ar^n + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{for} \quad |r| < 1.$$

If $|r| \geq 1$, the geometric series is diverges.

(4) Determine whether the following series are convergent or divergent. If a series is convergent, find its sum.

(a) $\sum\limits_{n=1}^{\infty} 5^{-n+1}3^n$.

(b) $\sum_{n=1}^{\infty} \frac{6}{n^2+3n}$.

(c) $8 - 10 + \frac{25}{2} - \frac{125}{8} + \frac{625}{32} - \cdots$

(d) $\sum_{n=1}^{\infty} \frac{n^3}{n(n+1)}$

(e) $\sum_{n=0}^{\infty} \frac{7 \cdot 2^{2n}}{\pi^{n+1}}$

(f) $\sum_{n=3}^{\infty} (e^{\frac{3}{n-2}} - e^{\frac{3}{n}})$

Test of Divergence

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(g) $\sum_{n=1}^{\infty} \ln\left(\dfrac{3e^{2n}}{e^{2n}+4n}\right)$

(5) Assume the nth partial sum of the series $\sum_{n=1}^{\infty} a_n$ is $s_n = \dfrac{3n+2}{1-2n}$.

(a) Find a_1.

(b) Find a formula for a_n when $n > 1$.

(c) Find $\sum_{n=1}^{\infty} a_n$.

(6) Let $a_n = \dfrac{2n^2}{5n^2 - 3n}$

(a) Determine whether $\{a_n\}$ is convergent.

(b) Determine whether $\sum_{n=1}^{\infty} a_n$ is convergent.

(7) Find the values of x of which the following series converge. Write your answers in interval notation. Find the sum of the series for those values of x.

(a) $\sum_{n=1}^{\infty} 2^{n-1}(x-3)^n$

(b) $\sum_{n=0}^{\infty} \frac{(-x)^n}{7^{n+1}}$