Note ♯2: Substitution and Area Between Curves

Problem 1. Compute $\int \frac{\cos^{-1}(3x)}{\sqrt{1-9x^2}} dx$.

Note that $\frac{d}{dx}\cos^{-1}(3x) = \frac{-3}{\sqrt{1-9x^2}}$ and so we let $u = \cos^{-1} 3x$. Then $du = -\frac{3dx}{\sqrt{1-9x^2}}$. Substituting in the integral:

$$\begin{aligned}\int \frac{\cos^{-1}(3x)}{\sqrt{1-9x^2}} dx &= \int \frac{u}{\sqrt{1-9x^2}} \frac{du\sqrt{1-9x^2}}{-3} \\ &= -\frac{1}{3}\int u du \\ &= -\frac{1}{6}u^2 + C \\ &= -\frac{1}{6}(\cos^{-1} 3x)^2 + C.\end{aligned}$$

Problem 2. Compute $\int e^{2x} \cos e^{2x} dx$.

Take $u = e^{2x}$ the $du = 2e^{2x}dx$ and the integral becomes:

$$\begin{aligned}\int e^{2x} \cos u \frac{du}{2e^{2x}} &= \frac{1}{2}\int \cos u du \\ &= \frac{1}{2}\sin u + C \\ &= \frac{1}{2}\sin e^{2x} + C.\end{aligned}$$

Problem 3. Compute $\int \frac{1}{\sin^2 x\sqrt{1-\cot x}}dx$.

Note that if $u = \cot x$ then $du = -(\csc x)^2 = \frac{-1}{\sin^2 x}dx$. Using this the integral becomes:

$$\begin{aligned}\int \frac{1}{\sin^2 x\sqrt{1-\cot x}}dx &= \int \frac{1}{\sin^2 x\sqrt{1-u}}(-\sin^2 x)dx \\ &= -\int \frac{1}{\sqrt{1-u}}du \\ &= 2\sqrt{1-u} + C \\ &= 2\sqrt{1-\cot x} + C.\end{aligned}$$

Problem 4. Compute $\int_{x=0}^{3} \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}dx$.

Observe that if $u = \sqrt{x+1}$ then $du = \frac{1}{2}\frac{1}{\sqrt{x+1}}$. Plugging this into the integral:

$$\begin{aligned}\int_{x=0}^{1} \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}dx &= \int_{u=1}^{u=2} \frac{e^u}{\sqrt{x+1}}2\sqrt{x+1}du \\ &= \int_{u=1}^{u=2} 2e^u du \\ &= 2e^2 - 2e \\ &= 2e(e-1).\end{aligned}$$

Problem 5. Compute $\int_0^1 x^7\sqrt{x^4+5}dx$.

This one is a little trickier. In my opinion, there isn't a substitution that just leaps out of the page. I think a good start though is $u = x^4+5$ and then we'll try to solve for the x^7 (or whatever ends up being there) in terms of u. So $du = 4x^3du$. Putting this into the integral:

$$\begin{aligned}
\int_{x=0}^{1} x^7\sqrt{x^4+5}dx &= \int_{u=5}^{6} x^7\sqrt{u}\frac{du}{4x^3} \\
&= \frac{1}{4}\int_{u=5}^{6} x^4\sqrt{u}du \\
&= \frac{1}{4}\int_{u=5}^{6} (u-5)\sqrt{u}du \\
&= \frac{1}{4}\int_{u=5}^{6} u^{3/2}-5u^{1/2}du \\
&= \frac{1}{4}\left(\frac{2}{5}u^{5/2}-\frac{10}{2}u^{3/2}\right)|_{u=5}^{6} \\
&= \frac{1}{4}\left(\frac{2}{5}6^{5/2}-\frac{10}{2}6^{3/2}\right)-\frac{1}{4}\left(\frac{2}{5}5^{5/2}-\frac{10}{2}5^{3/2}\right).
\end{aligned}$$

Problem 6. Compute $\int \frac{\sec\frac{1}{x^2}\tan\frac{1}{x^2}}{x^3}dx$.

Observe that if $u = \frac{1}{x^2}$ then $du = \frac{-2}{x^3}dx$. Using this in the integral:

$$\begin{aligned}
\int \frac{\sec\frac{1}{x^2}\tan\frac{1}{x^2}}{x^3}dx &= \int \frac{\sec u\tan u}{x^3}\frac{x^3du}{-2} \\
&= -\frac{1}{2}\int \sec u\tan udu \\
&= -\frac{1}{2}\sec u + C \\
&= -\frac{1}{2}\sec\frac{1}{x^2}+C.
\end{aligned}$$

Problem 7. Compute $\int_{x=0}^{2} \frac{x-x^3}{1+x^4} dx$.

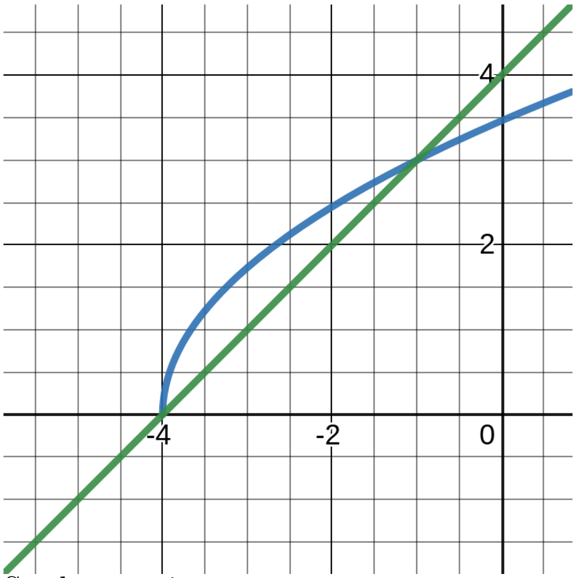
We will break this into two integrals:

$$\int_{x=0}^{2} \frac{x-x^3}{1+x^4} dx = \int_{x=0}^{2} \frac{x}{1+x^4} dx - \int_{x=0}^{2} \frac{x^3}{1+x^4} dx.$$

For the first integral, we will use $u = x^2$ then $du = 2x dx$. For the second intergal, we will take $w = 1 + x^4$ and $dw = 4x^3 dx$. This becomes:

$$\begin{aligned} \int_{x=0}^{2} \frac{x}{1+x^4} dx - \int_{x=0}^{2} \frac{x^3}{1+x^4} dx &= \frac{1}{2} \int_{u=0}^{4} \frac{du}{1+u^2} - \frac{1}{4} \int_{w=1}^{17} \frac{dw}{w} \\ &= \frac{1}{2} \arctan 4 - \frac{1}{4} \ln 17. \end{aligned}$$

Problem 8. Sketch the region bounded by $y = \sqrt{3x+12}$ and $y = x+4$ and find the area between them.

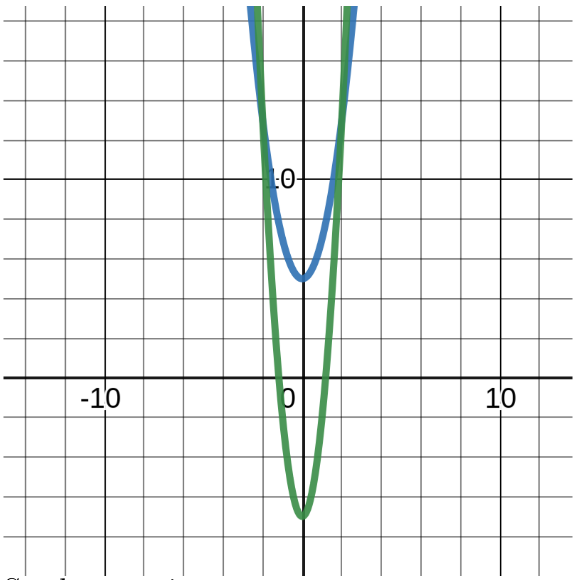


We need to find the intersection points: $\sqrt{3x+12} = x+4$. Solving this for x we get $3x+12 = x^2+8x+16$. This gives $x^2+5x+4 = (x+4)(x+1)$ so the intersection points are $x = -4$ and $x = -1$.

So the area is:

$$\begin{aligned}\int_{x=-4}^{-1}[(\sqrt{3x+12})-(x+4)]dx &= \int_{x=-4}^{-1}\sqrt{3x+12} - \int_{x=-3}^{-2}(x+4)dx \\ &= \frac{1}{3}\int_{u=0}^{9}\sqrt{u}du - \int_{x=-3}^{-2}(x+4)dx \\ &= 2(27) - \left((\frac{1}{2}-4)-(\frac{16}{2}-16)\right) \\ &= 49.5.\end{aligned}$$

Problem 9. Sketch the region bounded by $y = 2x^2 + 5$ and $y = 5x^2 - 7$ and find the area between them.

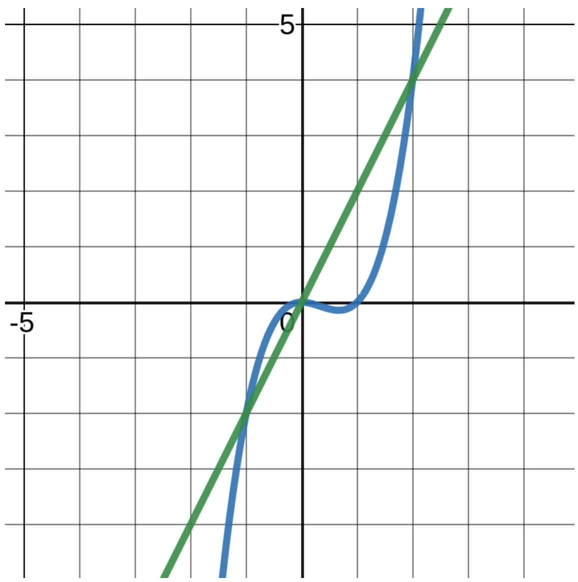


We need to find the intersection points: $2x^2 + 5 = 5x^2 - 7$ this gives the solutions $x = \pm 2$.

So the area is:

$$\int_{x=-2}^{2} [(2x^2+5)-(5x^2-7)]dx = 2\int_{x=0}^{2}(-3x^2+12)dx$$
$$= 40.$$

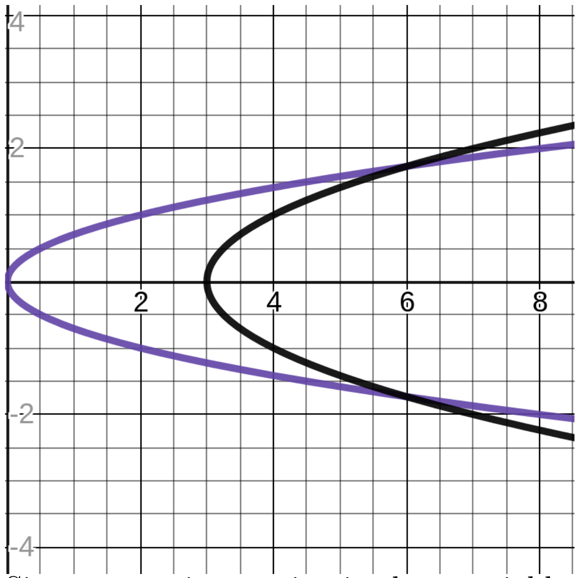
Problem 10. Sketch the region bounded by $y = x^3 - x^2$ and $y = 2x$ and find the area between them.



We need to find the intersection points: $x^3 - x^2 = 2x$ this gives $x = -1, 0, 2$. Note that on $[-1, 0]$ the the cubic function is on top and it swaps on the interval $[0, 2]$.

$$\int_{x=-1}^{0} [(x^3 - x^2) - (2x)]dx + \int_{x=0}^{2} [2x - (x^3 - x^2)]dx = \frac{15}{4}.$$

Problem 11. Sketch the region bounded by $x = 2y^2$ and $x = 3 + y^2$ and find the area between them.

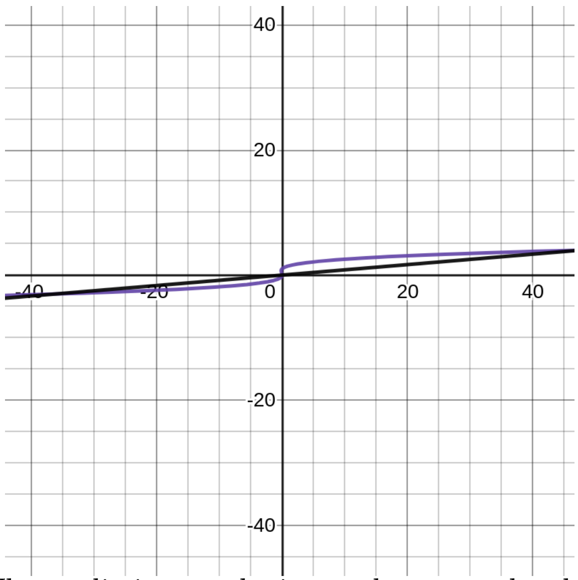


We need to find the intersection points: $2y^2 = 3 + y^2$ or $y^2 = 3$ so $y = \pm\sqrt{3}$.

Since we are integrating in the y variable, we do the rightmost function minus the leftmost.

$$\begin{aligned} A &= \int_{y=-\sqrt{3}}^{\sqrt{3}} [(3 + y^2) - (2y^2)]dy \\ &= 2\int_{y=0}^{\sqrt{3}} (-y^2 + 3)dy \\ &= 4\sqrt{3}. \end{aligned}$$

Problem 12. Sketch the region bounded by $x = y^3 - y^2$ and $x = 12y$ and find the area between them.

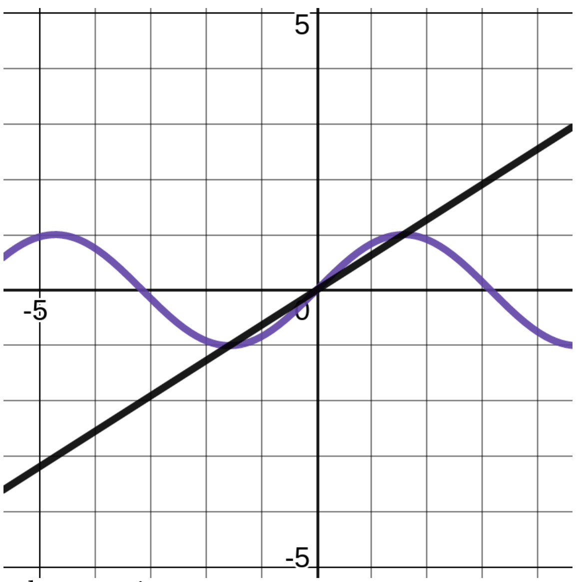


We need to find the intersection points: $y^3 - y^2 = 12y$ or $y(y^2 - y - 12) = y(y-4)(y+3)$ so the intersection points are $y = -3, 0, 4$.

When splitting up the intervals, we need to have the rightmost function minus the leftmost. This is:

$$\begin{aligned} A &= \int_{y=-3}^{0} [(y^3 - y^2) - (12y)]dy + \int_{y=0}^{4} [(12y) - (y^3 - y^2)]dy \\ &= -[\frac{81}{4} + 9 - 54] + [-64 + \frac{64}{4} + 6(16)] \\ &= \frac{291}{4}. \end{aligned}$$

Problem 13. Sketch the region bounded by $y = \sin x$ and $y = \frac{2}{\pi}x$ and find the area between them.

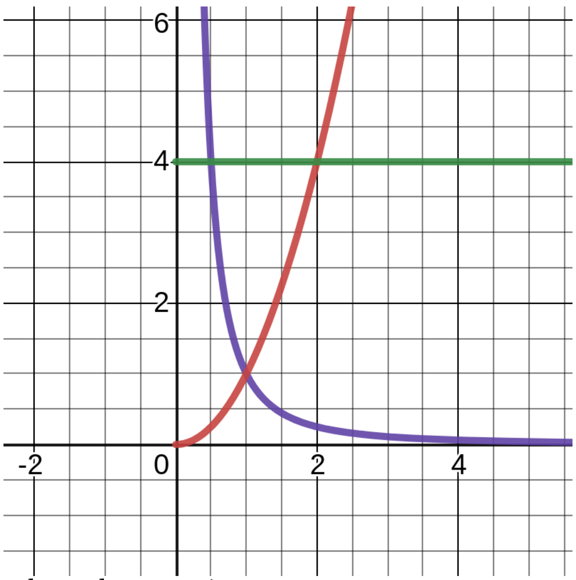


We need to find the intersection points: $\sin x = \frac{2}{\pi}x$. The solutions are $x = \pm\frac{\pi}{2}$ and $x = 0$.

So the area is:

$$\begin{aligned} A &= \int_{-\frac{\pi}{2}}^{0} [(\frac{2}{\pi}x) - (\sin x)]dx + \int_{x=0}^{\frac{\pi}{2}} [\sin x - \frac{2}{\pi}x]dx \\ &= 2\int_{x=0}^{\frac{\pi}{2}} [\sin x - \frac{2}{\pi}x]dx \\ &= 2\left(\cos x - \frac{2}{\pi}x\right)\Big|_{x=0}^{\frac{\pi}{x}} \\ &= 2 - \frac{\pi}{2}. \end{aligned}$$

Problem 14. Sketch the region bounded by $y = \frac{1}{x^2}$, $y = x^2$ and $y = 4$ and in the first quadrant. Find the area of this region.



It's better here to view this as a y integral. So the first function is $x = \frac{1}{\sqrt{y}}$ and the second one is $x = \sqrt{y}$. To find the intersection points, $\frac{1}{\sqrt{y}} = \sqrt{y}$ gives $y = 1$.

So then the area is:

$$A = \int_{y=1}^{4} (\sqrt{y} - \frac{1}{\sqrt{y}})dy$$
$$= \frac{20}{3}.$$