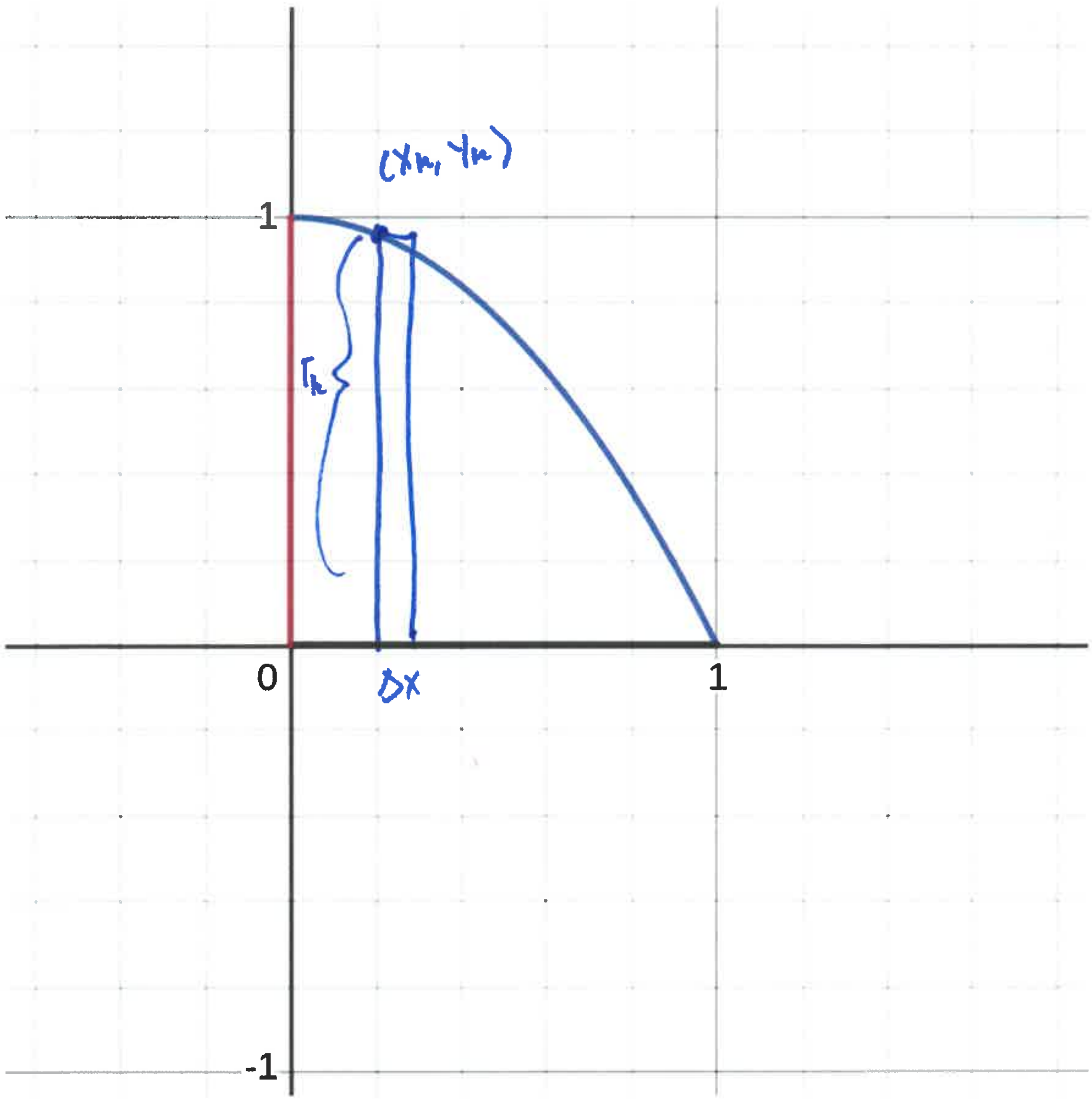
NOTE ♯2: VOLUMES (DISKS AND WASHERS)

Problem 1. Find the volume of the solid obtained by rotating the region bounded by $y = -x^2+1$, $y = 0$; $x = 0$; about the x axis.



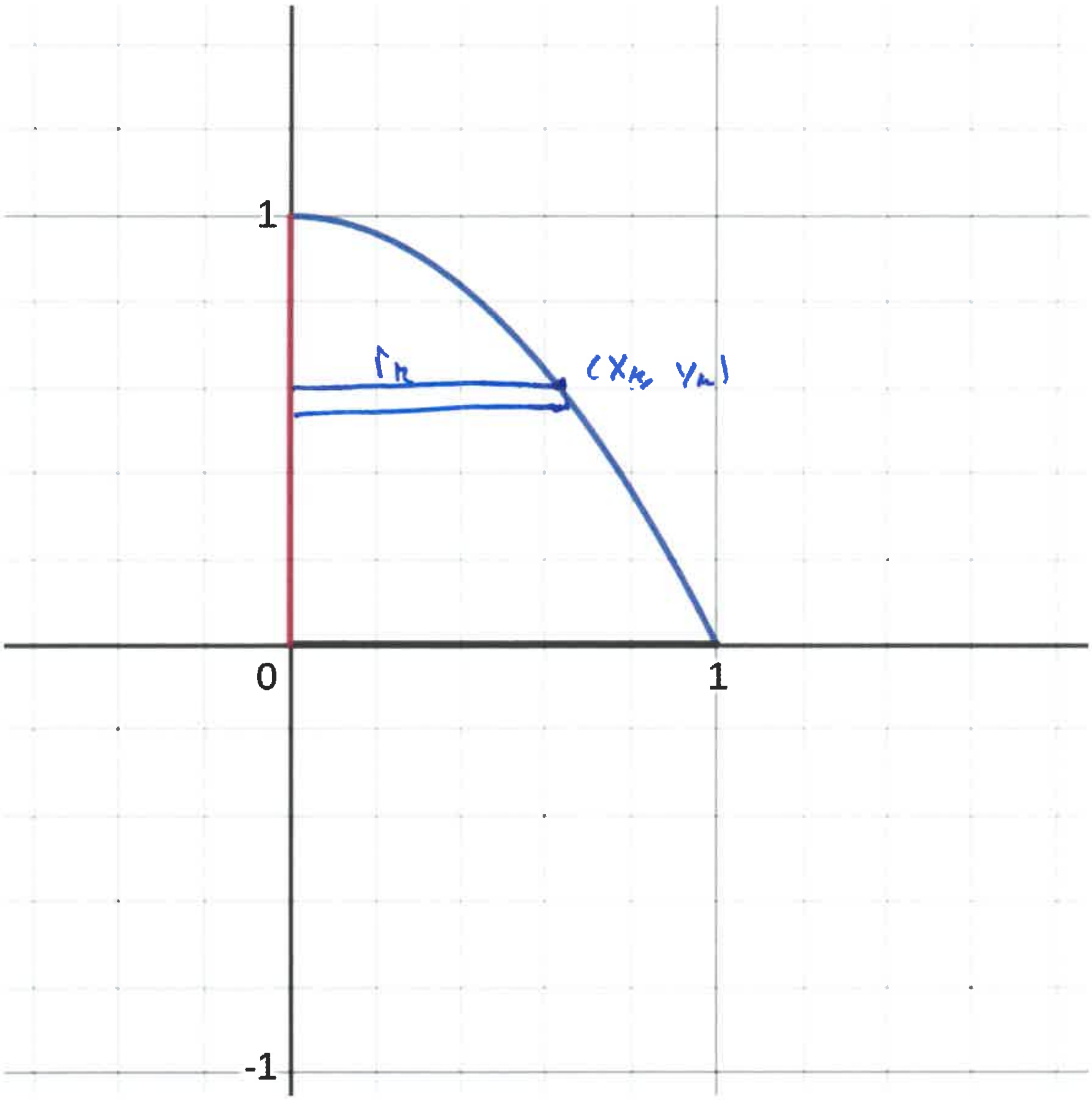
We will rotate a rectangle going from the x axis to the point (x_k, y_k) on the purple curve. This will make a disk when rotated and the volume of a disk is its area times its thickness:

$$\Delta V_k \simeq \pi r_k^2 \Delta x = \pi y_k^2 \Delta x = \pi(-x_k^2 + 1)^2 \Delta x.$$

Then the volume is:

$$V = \int_{x=0}^{1} \pi(-x^2 + 1)^2 dx = \frac{8}{15}\pi.$$

Problem 2. Find the volume of the solid obtained by rotating the region bounded by $y = -x^2+1$, $y = 0$; $x = 0$; about the y axis.



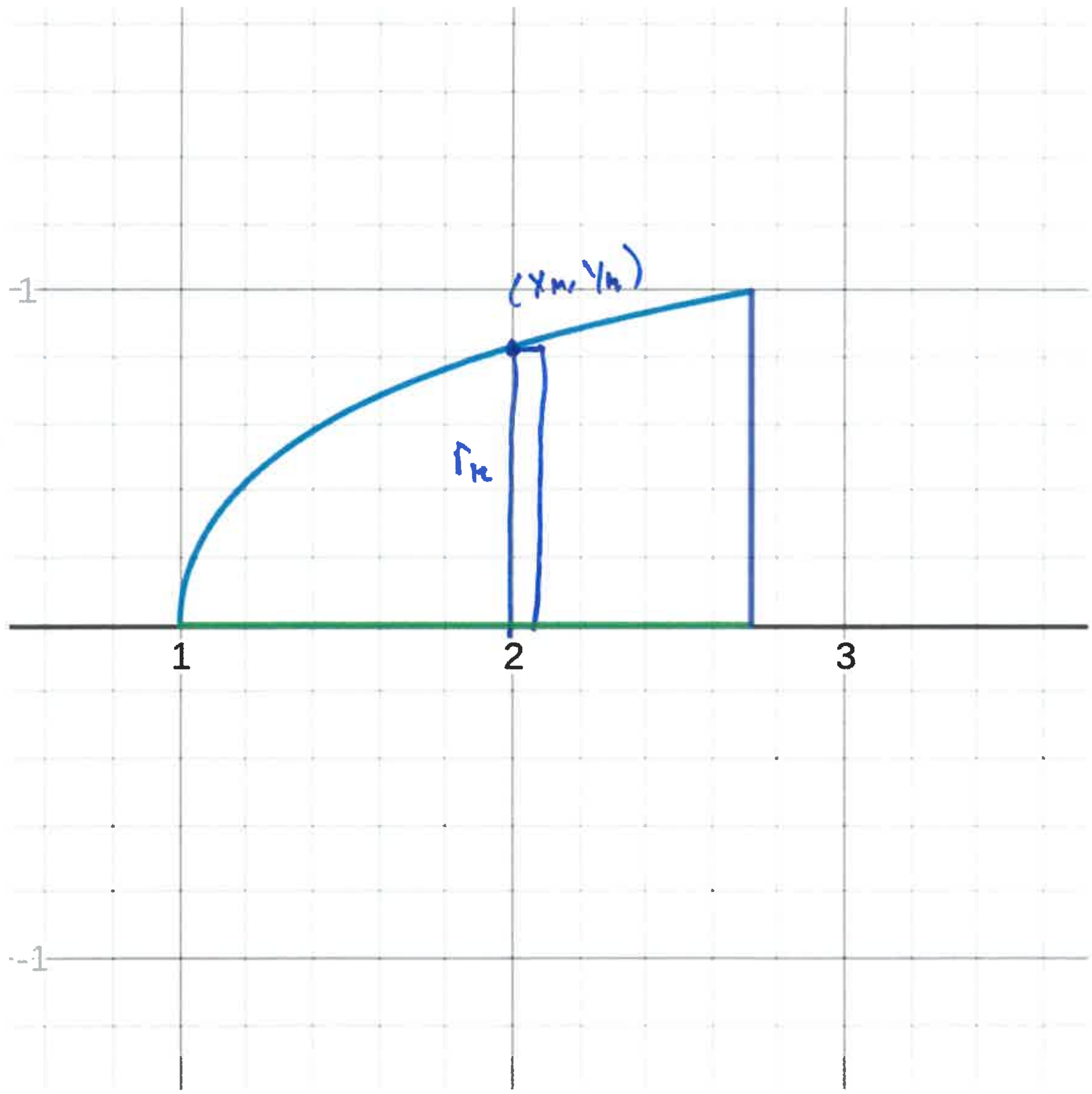
This is similar to the previous problem (the region is the same, that is). We will rotate a horizontal rectangle around the y-axis. The rectangle will go from the y axis to a point (x_k, y_k) on the purple curve. The volume of this rectangle when rotated around the y axis is:

$$\Delta V_k \simeq \pi r_k^2 \Delta y = \pi x_k^2 \Delta y = \pi(1 - y_k)\Delta y.$$

Above, we used the fact that if (x_k, y_k) is on the purple curve, then $y_k = -x_k^2 + 1$. So the volume is:

$$V = \int_{y=0}^{1} \pi(1 - y)dy = \frac{1}{2}\pi.$$

Problem 3. Find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{\ln x}$, $y = 0$; $x = e$; about the x axis (you may use the fact that $\int \ln x dx = x \ln x - x + C$).



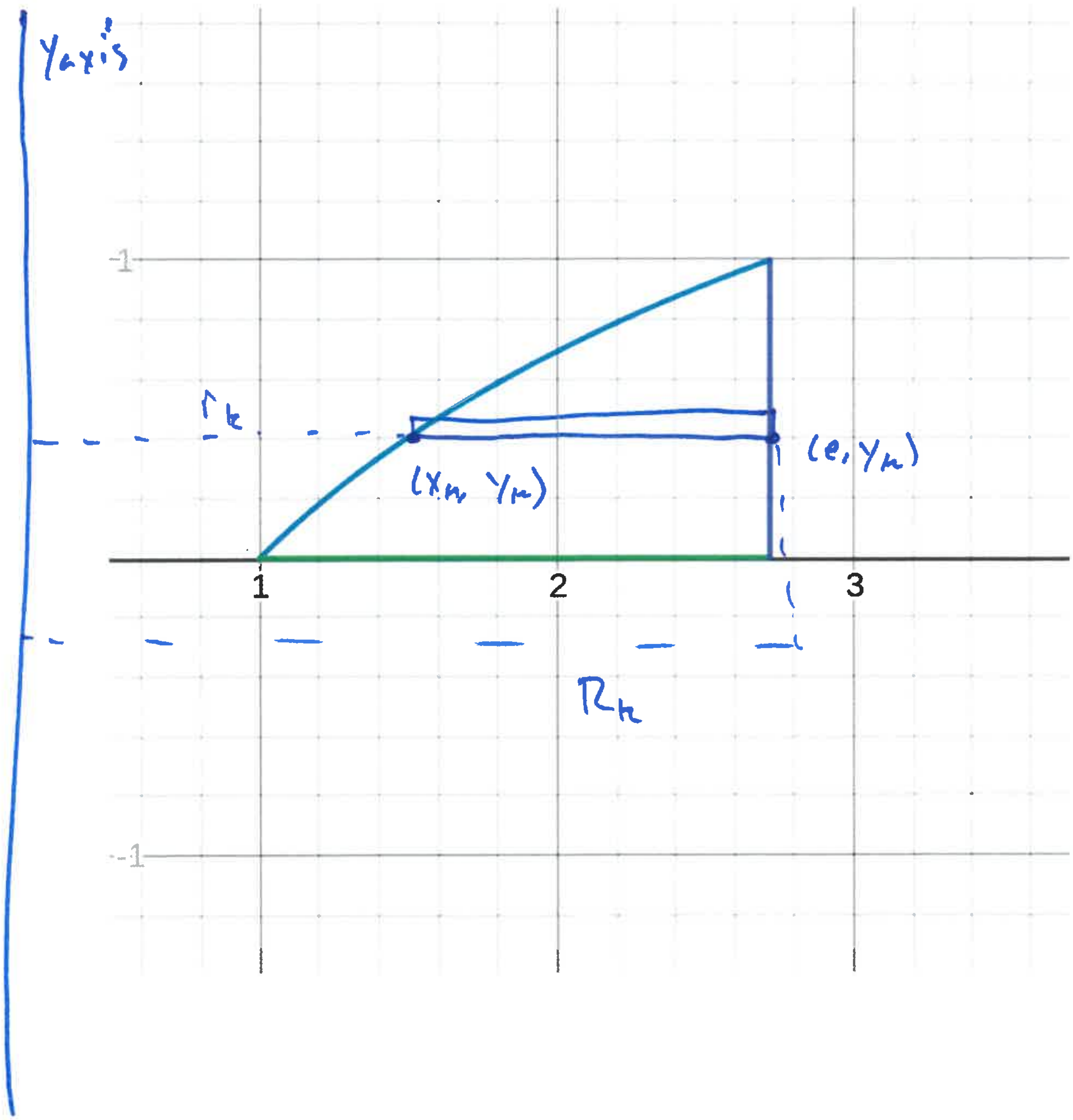
We will rotate a rectangle going from the x axis to a point (x_k, y_k) on the blue curve. The volume of this rectangle when roatated is:

$$\Delta V_k \simeq \pi r_k^2 \Delta x = \pi y_k^2 \Delta x = \pi \ln x \Delta x.$$

So then the volume is:

$$V = \int_{x=1}^{e} \pi \ln x dx = \pi.$$

Problem 4. Find the volume of the solid obtained by rotating the region bounded by $y = \ln x$, $y = 0$; $x = e$; about the y axis.



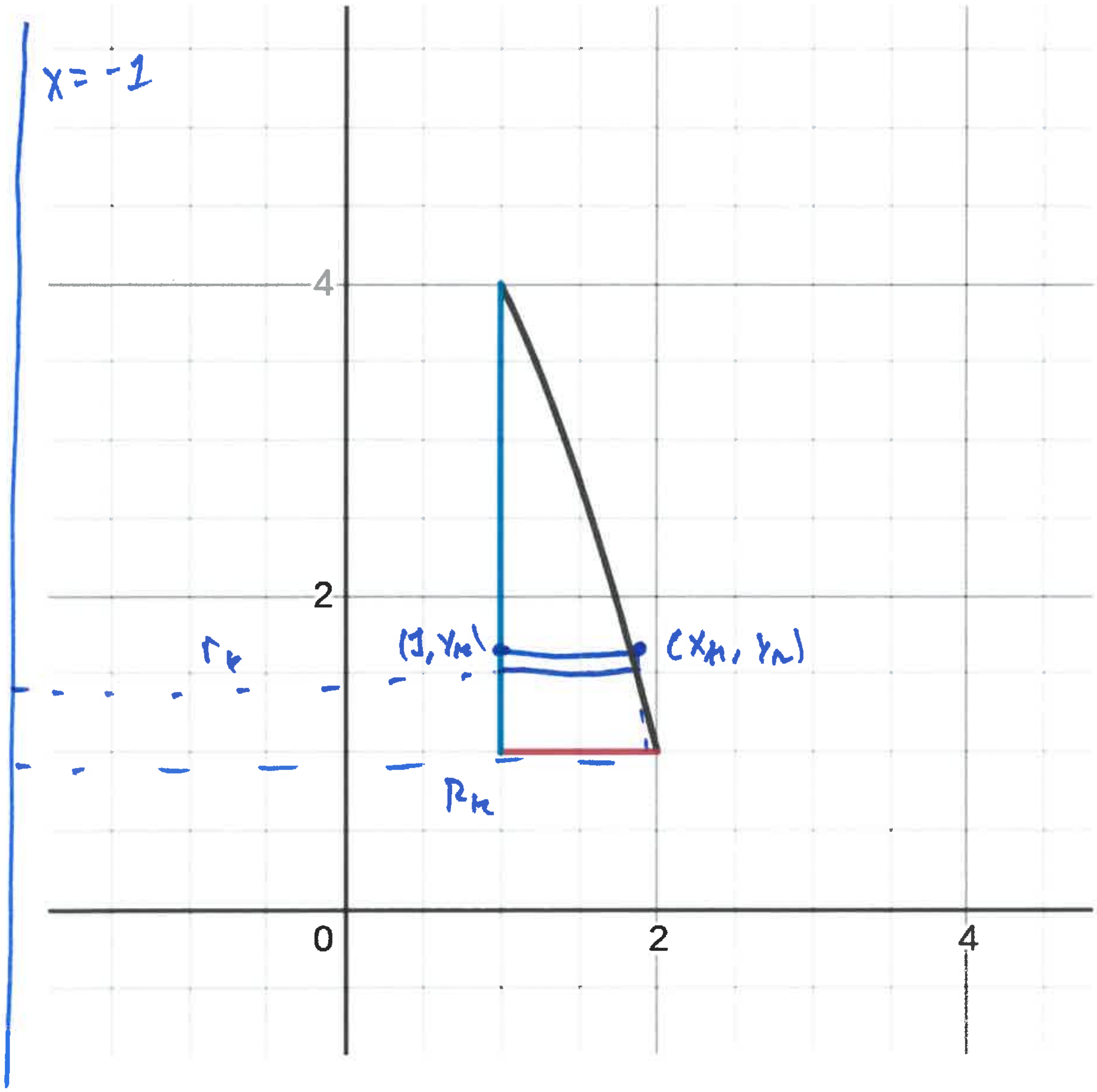
The region is similar to the one above. Only now around the y axis. So we use a horizontal rectangle going from a point (x_k, y_k) on the blue curve to (e, y_k) on the purple curve. Note that when this is rotated about the y axis, it makes a washer. The volume of a washer is $\pi(R_k^2 - r_k^2)T$ where T is thickness (in this case $T = \Delta y$). So then:

$$\begin{aligned}\Delta V_k &\simeq \pi(R_k^2 - r_k^2)\Delta y \\ &= \pi(e^2 - x_k^2)\Delta y \\ &= \pi(e^2 - e^{2y_k})\Delta y.\end{aligned}$$

So then the volume is:

$$V = \int_{y=0}^{1} \pi(e^2 - e^{2y})dy = \frac{1+e^2}{2}\pi.$$

Problem 5. Find the volume of the solid obtained by rotating the region bounded by $y = -x^2+5$, $y = 1$; $x = 1$; about the line $x = -1$.



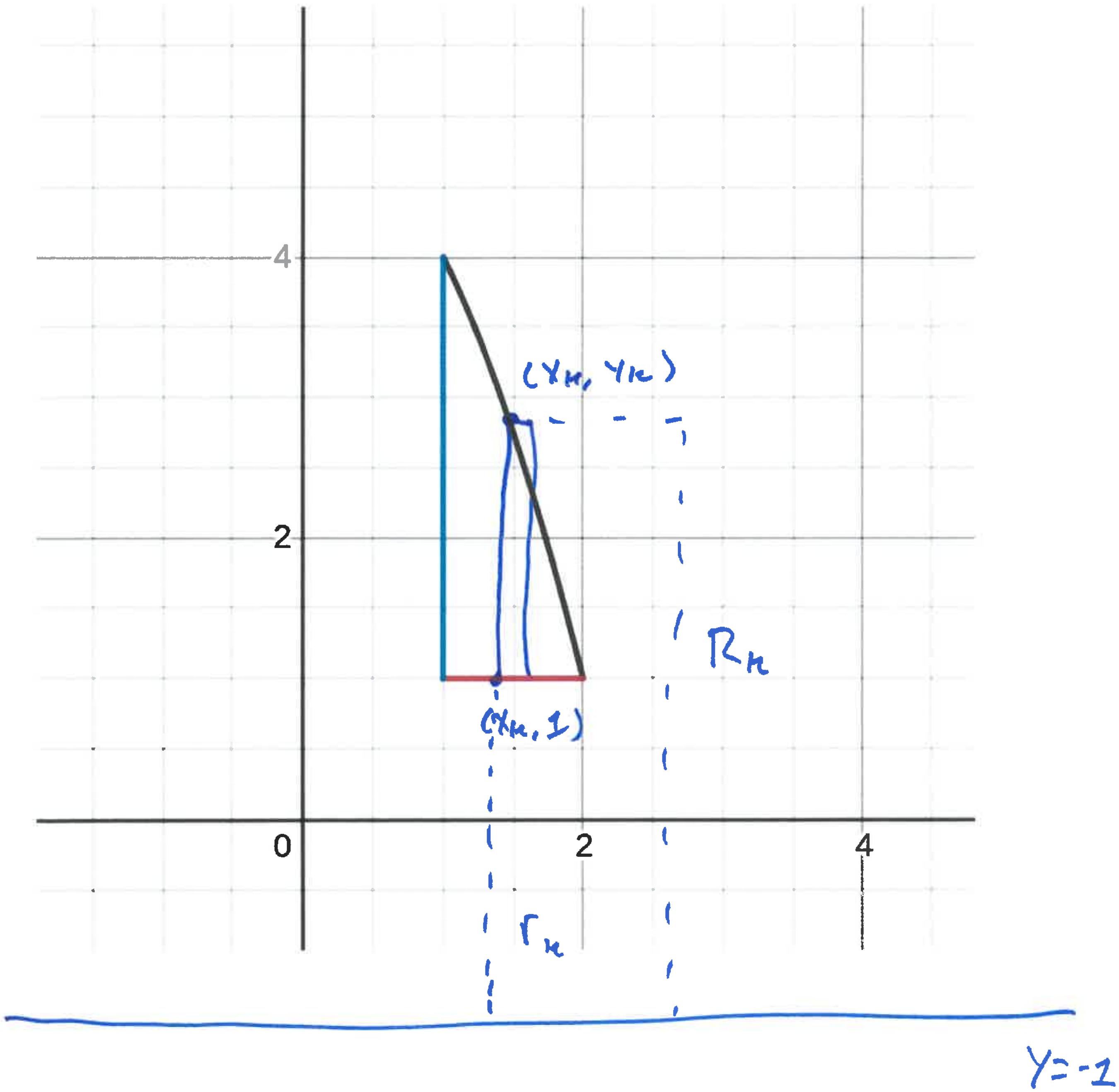
We will draw a horizontal rectangle from the point $(1, y_k)$ on the blue line to (x_k, y_k) on the black line. When that is rotated around the line $x = -1$ it forms a washer. So:

$$\begin{aligned}\Delta V_k &\simeq \pi(R_k^2 - r_k^2)\Delta y \\ &= \pi((x_k+1)^2 - (1+1)^2)\Delta y \\ &= \pi((\sqrt{5-y_k}+1)^2 - 4)\Delta y.\end{aligned}$$

So then the volume is:

$$V = \int_{y=1}^{4} \pi((\sqrt{5-y}+1)^2 - 4)dy = \frac{47}{6}\pi.$$

Problem 6. Find the volume of the solid obtained by rotating the region bounded by $y = -x^2+5$, $y = 1$; $x = 1$; about the line $y = -1$.



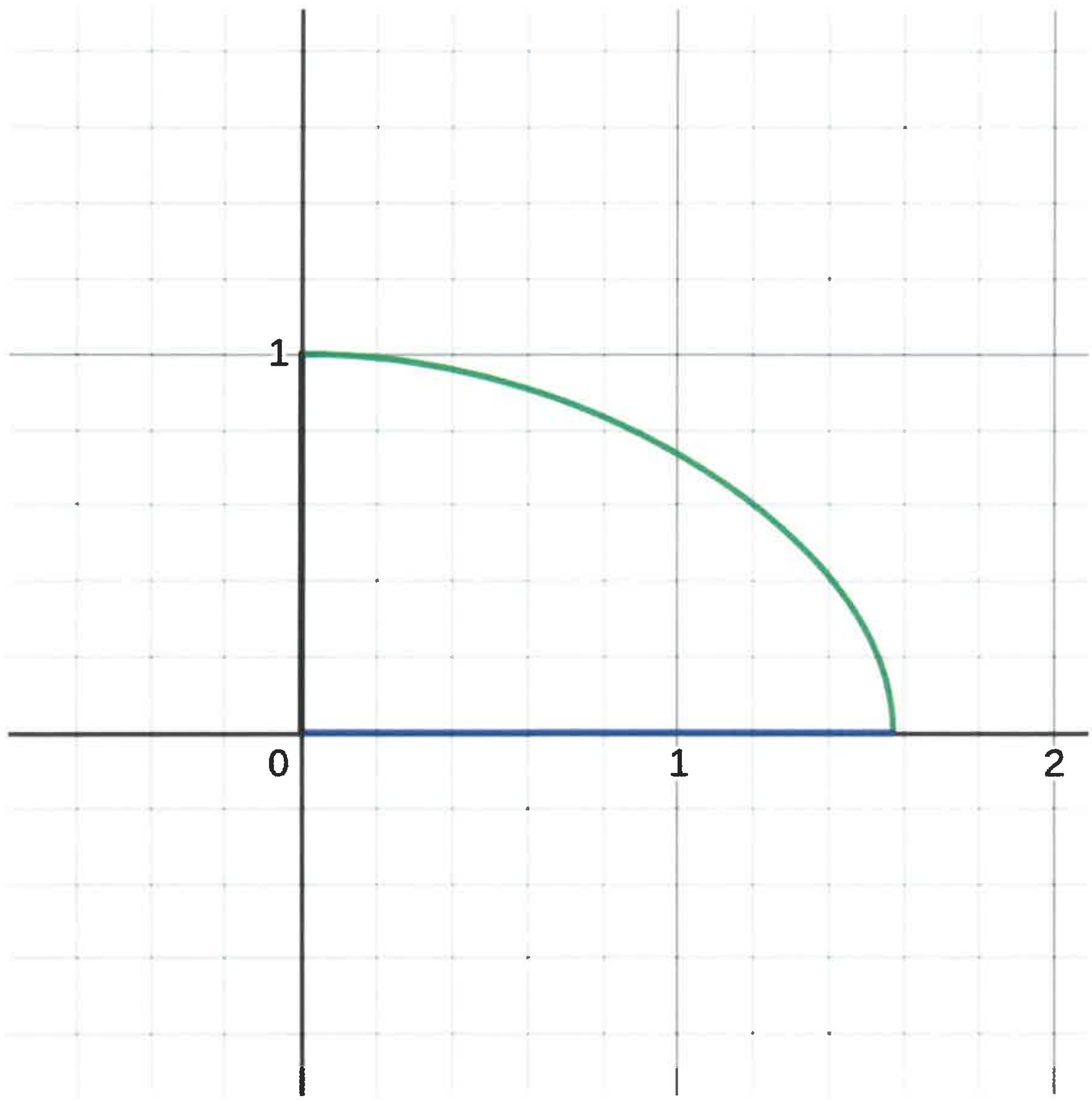
For this one, we will do a vertical rectangle from a point $(x_k, 1)$ on the redline to a point (x_k, y_k) on the black. When this is rotated, we get a washer. And its volume is:

$$\begin{aligned}\Delta V_k &\simeq \pi(R_k^2 - r_k^2)\Delta x\\ &= \pi[(y_k + 1)^2 - (1+1)^2]\Delta x\\ &= \pi[(-x_k^2 + 6)^2 - 4]\Delta x.\end{aligned}$$

So the volume is:

$$V = \int_{x=1}^{2} \pi[(-x^2+6)^2 - 4]dx = \frac{51}{5}\pi.$$

Problem 7. Find the volume of the solid whose base is the region bounded by $y = \sqrt{\cos x}$, $y = 0$, $x = 0$ and whose cross sections perpendicular to the x axis are semi–circles.



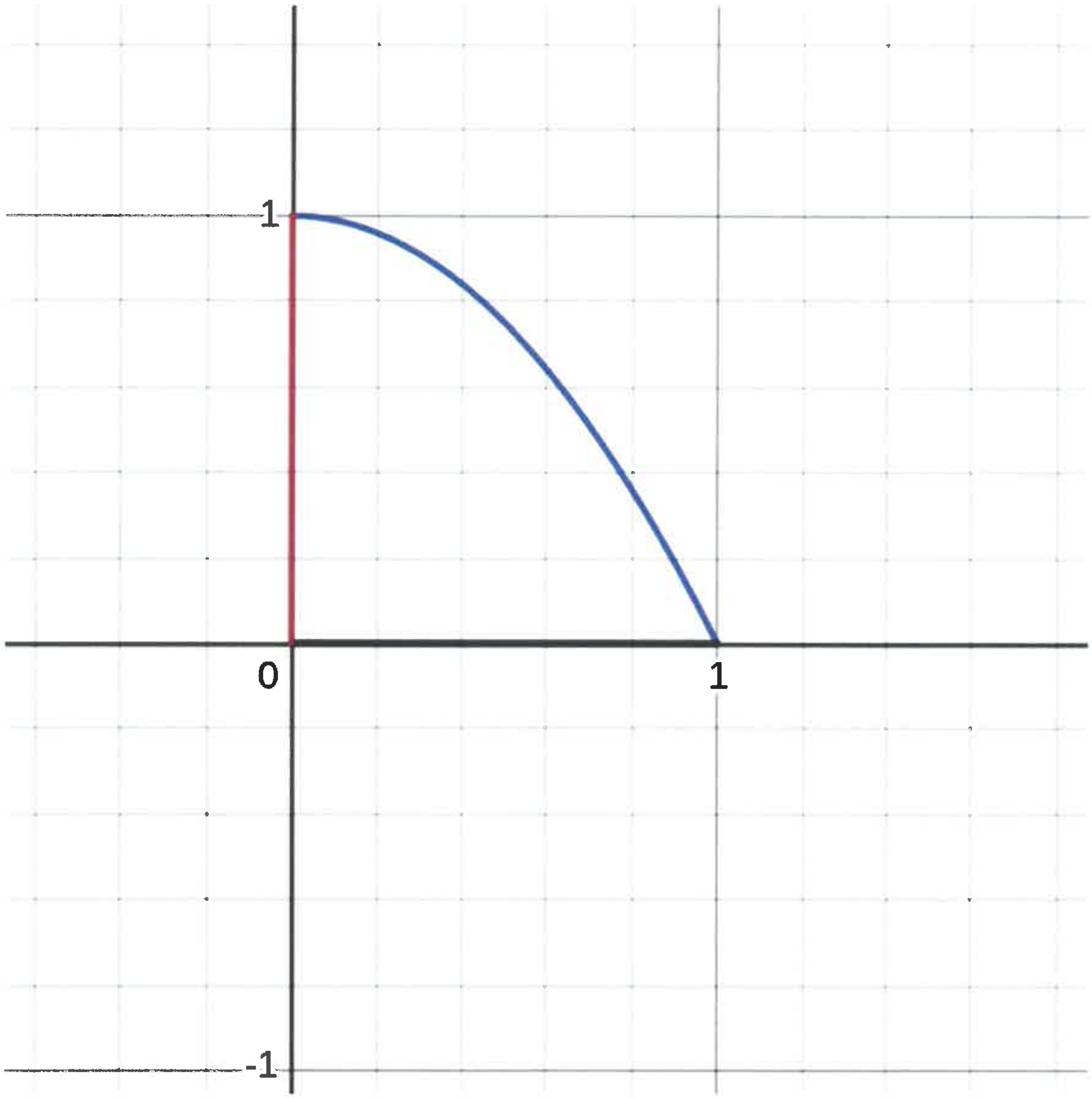
The base of the semi–circles go from the purple curve to the green curve. The radius of those circles are $\frac{1}{2}\sqrt{\cos x_k}$. So their volume is:

$$\Delta V_k \simeq \pi r_k^2 \Delta x = \pi \frac{\cos x_k}{4} \Delta x.$$

So the volume is:

$$V = \int_{y=0}^{\pi/2} \frac{\pi}{4} \cos x dx = \frac{\pi}{4}.$$

Problem 8. Find the volume of the solid whose base is the region bounded by $y = -x^2 + 1$, $y = 0$, $x = 0$ and whose cross sections perpendicular to the y axis are squares.



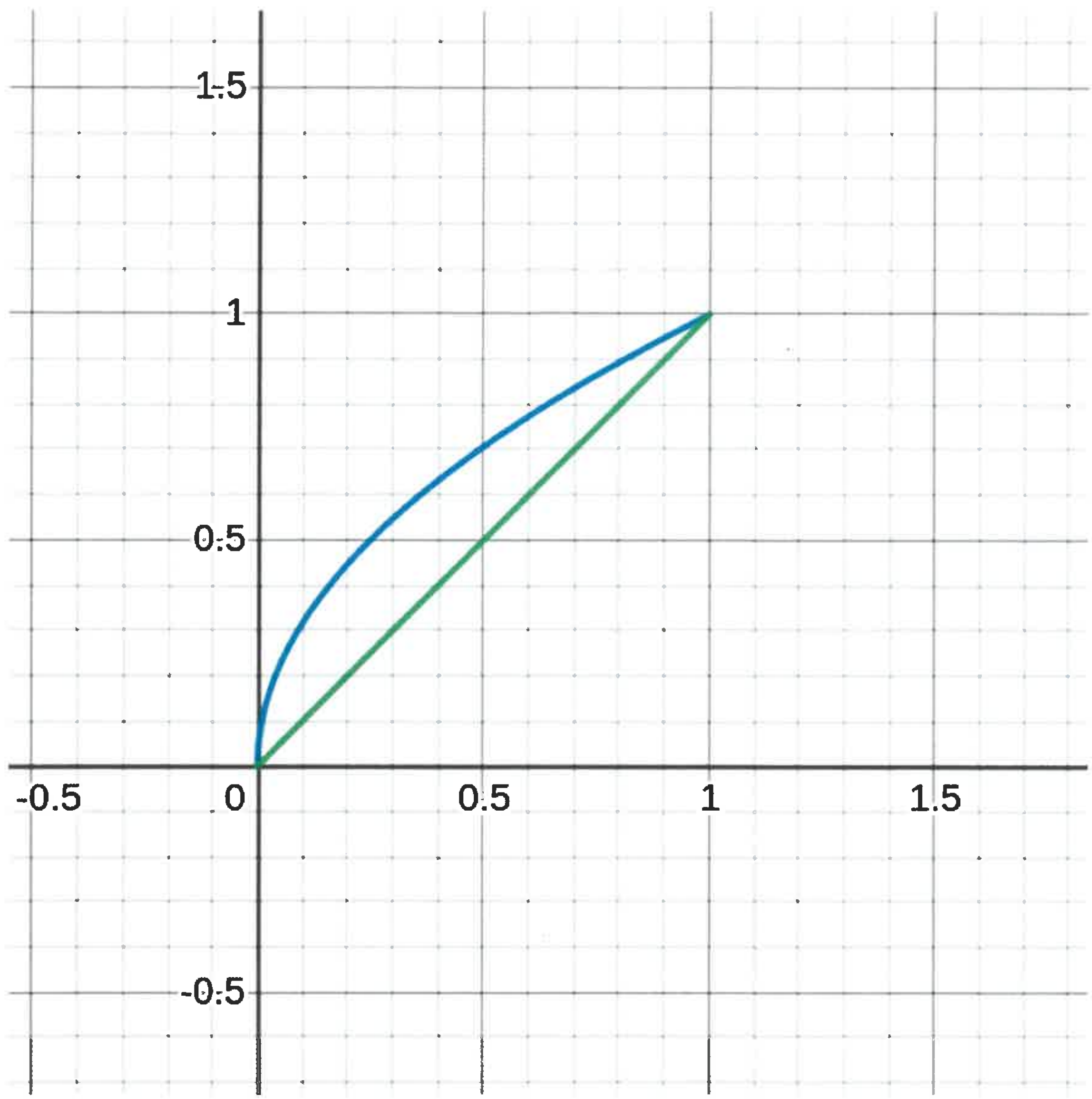
The squares have bases going from the y axis to the purple curve. So their side length is $x_k = \sqrt{1-y_k}$ and so their volume is:

$$\Delta V_k \simeq (1-y_k)\Delta y.$$

So the volume of the solid is:

$$V = \int_0^1 (1-y)dy = \frac{1}{2}.$$

Problem 9. Find the volume of the solid whose base is the region bounded by $y = \sqrt{x}$, $y = x$ and whose cross sections perpendicular to the y axis are rectangles with height equal to twice the base.



The base of the rectangle goes from blue to green curve. So the base is equal to $(y_k - y_k^2)$. So then the area of the rectangle is $2(y_k - y_k^2)^2$. So the volume is:

$$V = \int_{x=0}^{2} 2(y - y^2)^2 dy = \frac{1}{30}.$$

Problem 10. Find the volume of the solid whose base is an equilateral triangle of side length one and whose cross sections parallel to one of the sides are squares.