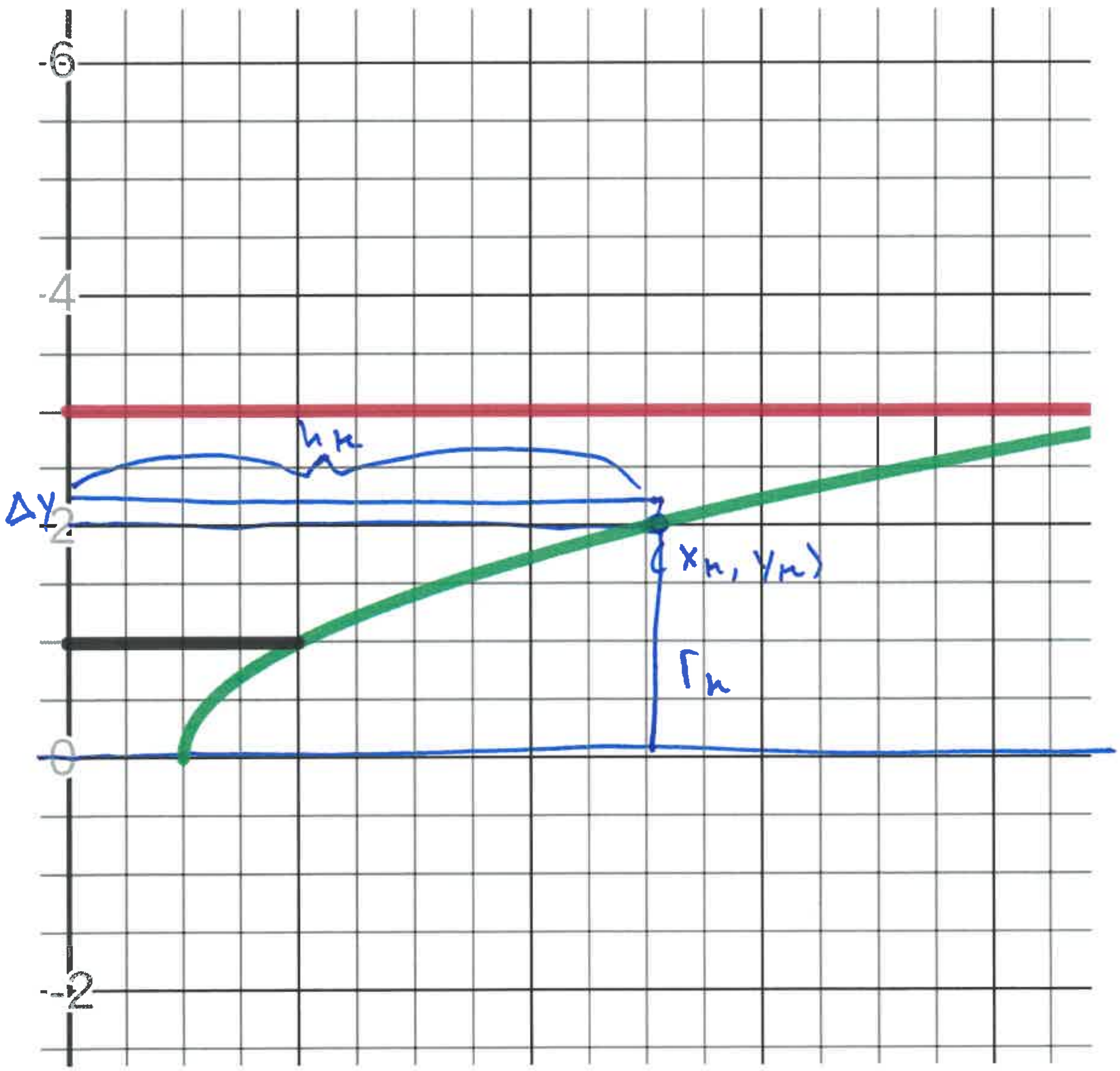
NOTE ♯2: VOLUMES (SHELLS)

Problem 1. Find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x-1}$, $y = 1$, $y = 3$ and the y–axis around the x–axis.



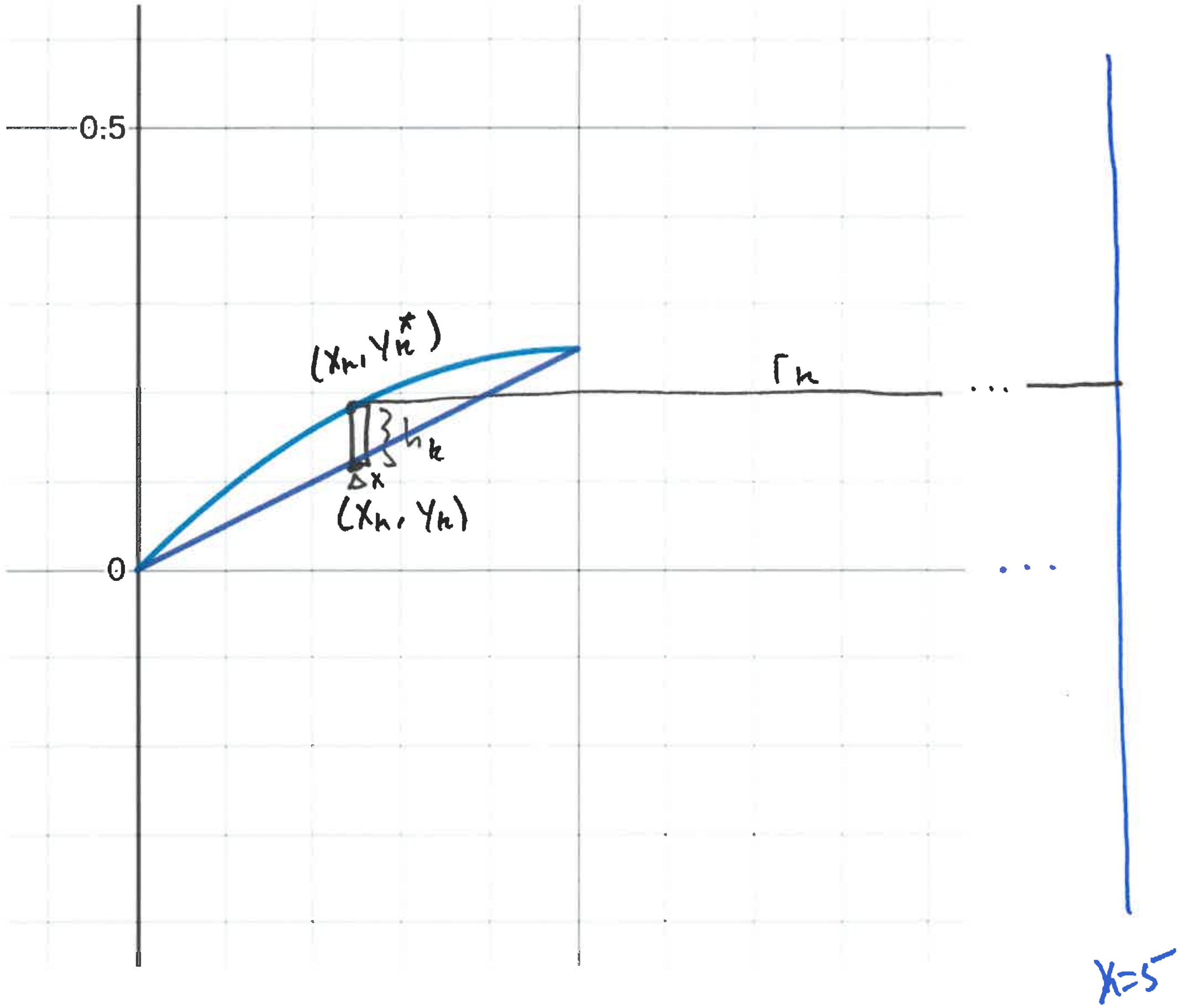
To use shells, we consider a horizontal rectangle going from the y–axis to a point (x_k, y_k) on the green curve. Then when we rotate that we get a shell and the volume is:

$$\Delta V_k \simeq 2\pi r_k h_k \Delta y = 2\pi(y_k)(x_k)\Delta y = 2\pi(y_k)(y_k^2 + 1)\Delta y.$$

To find the volume, we need to know the lowest and highest y values. The lowest is $y = 1$ and the highest is $y = 3$. So the volume is:

$$V = \int_{y=1}^{3} 2\pi y(y^2 + 1)dy = 48\pi.$$

Problem 2. Find the volume of the solid obtained by rotating the region bounded by $y = -x^2 + x$, $y = x/2$ around the line $x = 5$. (in the figure below, the line $x = 5$ isn't shown).



To use shells, consider a vertical rectangle going from the point (x_k, y_k*) on the blue curve to the point (x_k, y_k) on the purple curve. The volume of this piece is:

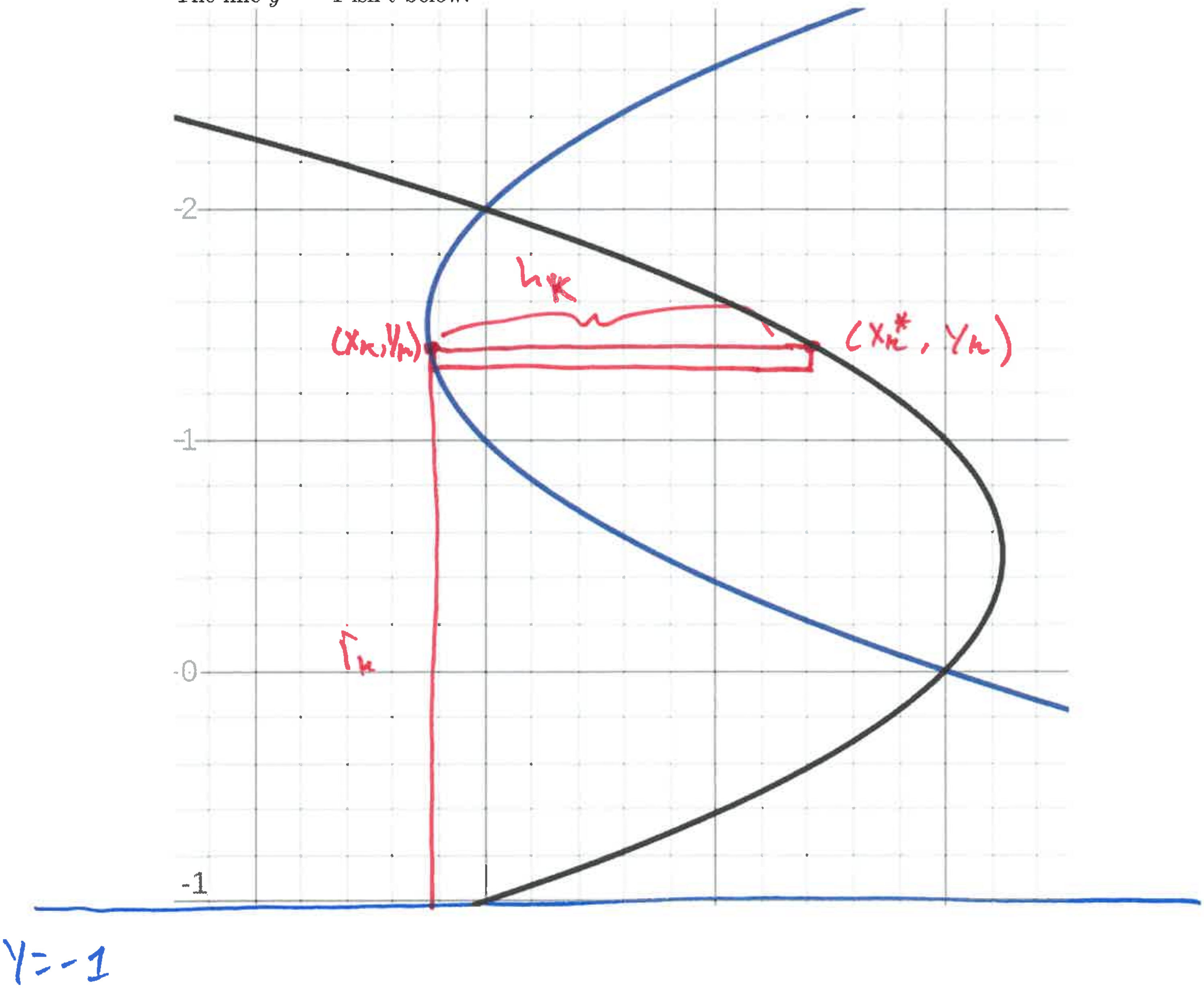
$$\begin{aligned}\Delta V_k \simeq 2\pi r_k h_k \Delta x &= 2\pi(5 - x_k)(y_k * - y_k)\Delta x \\ &= 2\pi(5 - x_k)((-x_k^2 + x_k) - \frac{x_k}{2})\Delta x \\ &= 2\pi(5 - x_k)(-x_k^2 + \frac{x_k}{2})\Delta x.\end{aligned}$$

To determine the intersection, solve $-x^2 + x = \frac{x}{2}$ to get $x = 0, 1/2$ so the volume is:

$$V = \int_{x=0}^{\frac{1}{2}} 2\pi(5 - x)(-x^2 + \frac{x}{2})dx = \frac{38}{192}\pi.$$

Problem 3. Find the volume of the solid obtained by rotating the region bounded by $x = y^2 - 3y + 4$ and $x = -y^2 + y + 4$ about the line $y = -1$.

The line $y = -1$ isn't below.



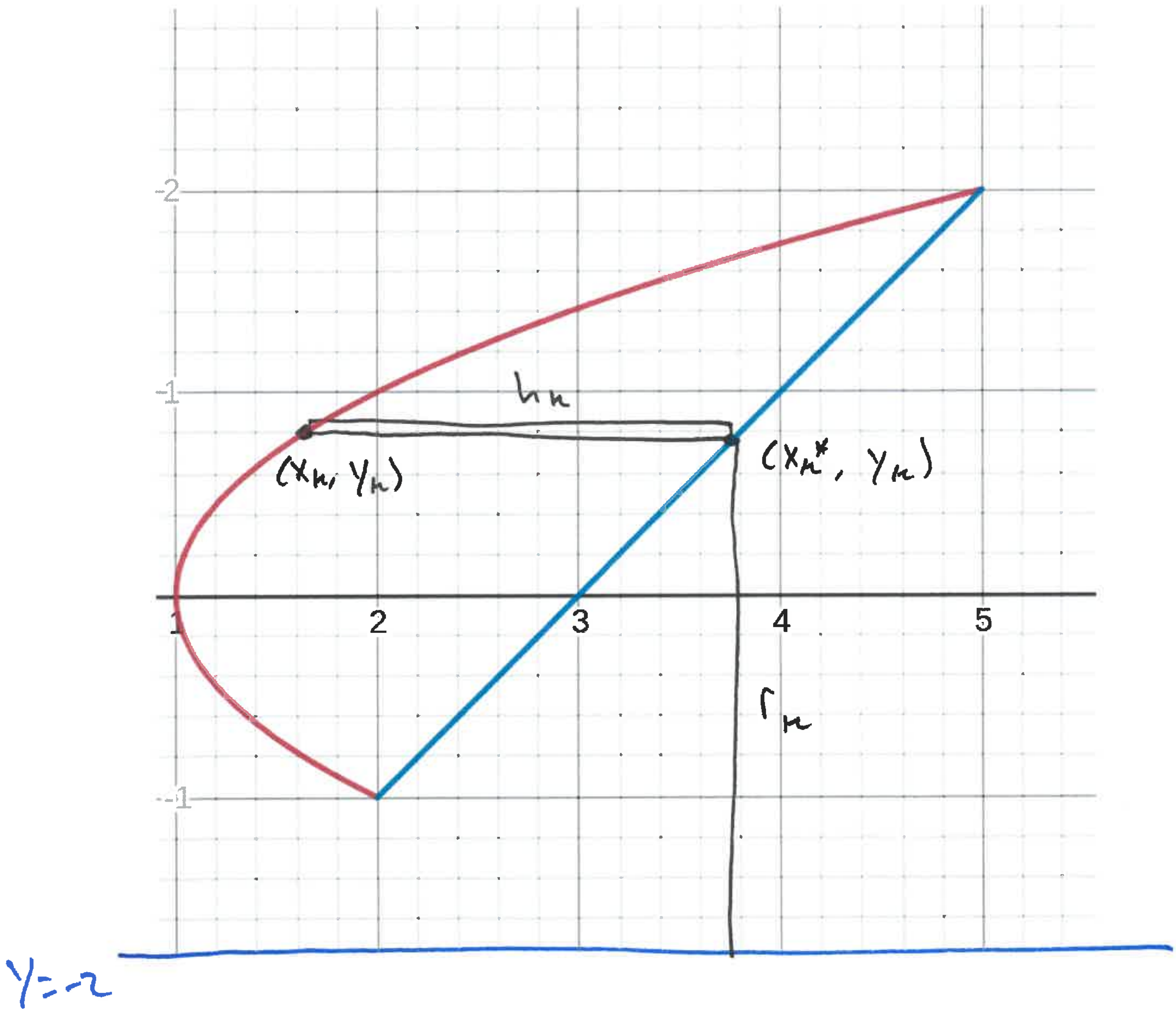
To use the shell method, we will consider a rectangle going from the point (x_k, y_k) and the purple curve to (x_k*, y_k) on the black curve. The volume of this rectangle when rotated around the line $y = -1$ is:

$$\begin{aligned}\Delta V_k &\simeq 2\pi r_k h_k \Delta y \\ &= 2\pi(y_k+1)(x_k * - x_k)\Delta y \\ &= 2\pi(y_k+1)[(-y_k^2+y_k+4)-(y_k^2-3y_k+4)]\Delta y.\end{aligned}$$

Now, to find the limits of integration, solve $y^2 - 3y + 4 = -y^2 + y + 4$ to get $y = 0$ and $y = 4$ so the volume is:

$$\begin{aligned}V &= \int_{y=0}^{2} 2\pi(y+1)[(-y^2+y+4)-(y^2-3y+4)]dy \\ &= \int_{y=0}^{2} 2\pi(y+1)(-2y^2+4y)dy \\ &= \frac{32\pi}{3}.\end{aligned}$$

Problem 4. Find the volume of the solid obtained by rotating the region bounded by $x = y^2 + 1$ and $x = y + 3$ about the line $y = -2$.



To use shells, we will consider a rectangle going from the point (x_k, y_k) on the red curve to the point (x_k*, y_k) on the blue curve. The volume of this rectangle when rotated around the line $y = -2$ is:

$$\begin{aligned}\Delta V_k &\simeq 2\pi r_k h_k \Delta y \\ &= 2\pi(y_k + 2)(x_k * - x_k)\Delta y \\ &= 2\pi(y_k + 2)(y_k + 3 - (y_k^2 + 1))\Delta y.\end{aligned}$$

To find the points of intersection, solve $y^2 + 1 = y + 3$ to get $y = -1$ and $y = 2$. So the volume is:

$$V = \int_{y=-1}^{2} 2\pi(y + 2)(-y^2 + y + 2)dy = \frac{45}{2}\pi.$$

NOTE ♯2: WORK

Problem 5. A particle is moved along the x axis by a force that measures $f(x) = x^3\sqrt{x^4+1}$ pounds at a point x feet from the origin. Find the work in moving the object from $x = 2$ to $x = 5$.

This is just:

$$W = \int_{x=2}^{5} x^3\sqrt{x^4+1}dx.$$

To compute this integral, let $u = x^4 + 1$. Then $du = 4x^3dx$ and the integral becomes:

$$W = \int_{x=2}^{5} x^3\sqrt{x^4+1}dx = \int_{17}^{626} u^{1/2}\frac{du}{4} = \frac{1}{6}u^{3/2}|_{17}^{625} = \frac{1}{6}(626^{3/2} - 17^{3/2}) \simeq 2598.7.$$

Problem 6. Suppose a spring has a natural length of .1 m and it takes a force of 8 N to hold it stretched to .3 m. How much work is required to stretch it from .15m to .25m?

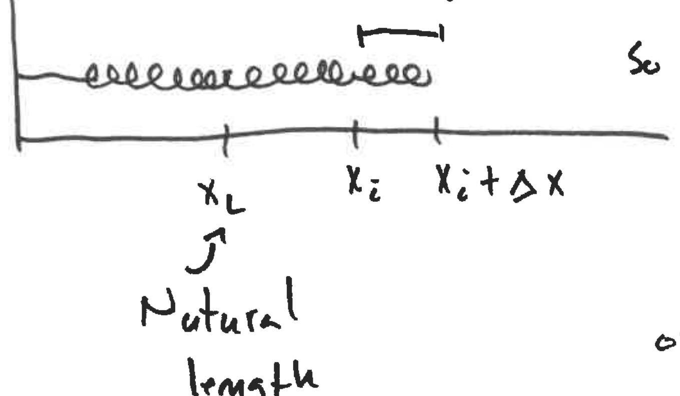
First, we can solve for the spring constant k using the equation $F = kx$ with $F = 8$ and $x = .2$ to get $k = 40$. Now, the force required to move the spring from x_i to $x_i + \Delta x$ is approximately:

$$\Delta W_k \simeq (k(x_i - .1))\Delta x = 40(x_i - .1)\Delta x.$$

So the force required to move the spring from .15 to .25 is:

$$W = \int_{.15}^{.25} 40(x - .1)dx = .4NM.$$

$$\Delta W_i \sim k(x_i - x_L)\,\Delta x$$



So work to go from x_0 to x_f is approx

$$W \approx \sum k(x_i - x_L)\Delta x$$

or, as $\Delta x \to 0$, this goes to:

$$W = \int_{x_0}^{x_f} k(x - x_L)\,dx.$$

Problem 7. A rope that is 30 m long and weighs 10 kg hangs over the top of a building. How much work does it take to pull the rope to the top? How much work does it take to pull the half the rope to the top?

First, we need the linear density. Since the rope weighs 10 kg and is 30m long, this means the linear density is $10/30 = 1/3$ (kg/m). We'll use a coordinate system where the 0 is the top of the building and 30 is the bottom. The work done to move a small section of rope that is x_k meters down from the top is

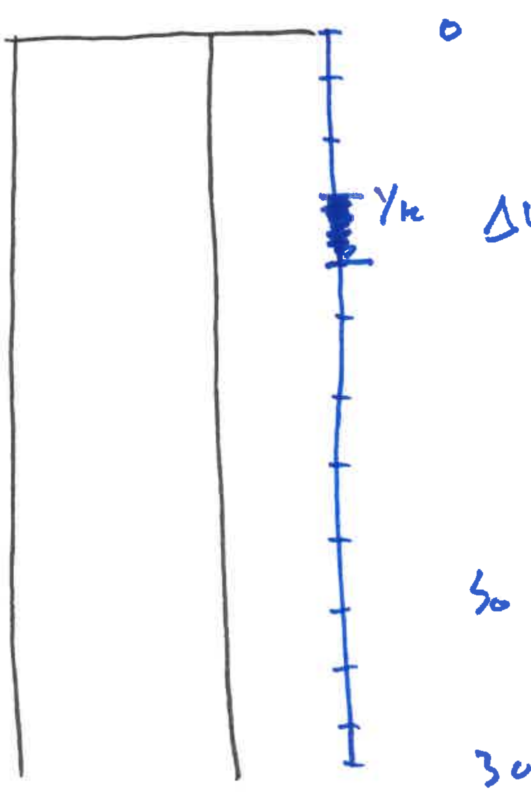
$$\Delta W_k \simeq 9.8(\frac{1}{3})\Delta x(x_k).$$

(Note: the "9.8" is acceleration due to gravity - i.e. weight = gm). So then the work done to pull the whole rope up is:

$$W = \int_0^{30} \frac{9.8}{3} x dx = \frac{9.8}{6} 30^2 NM.$$

The work done to pull half the rope is:

$$W = \int_0^{15} \frac{9.8}{3} x dx = \frac{9.8}{6} 15^2 NM.$$



$\Delta W_k \approx$ (weight) Y_k

$\approx g\,\delta\,\Delta Y\,Y_k$

δ ↑ linear mass density of rope $(=1/3)$

So $\Delta W_k \approx 9.8\,\frac{1}{3}\,Y_k\,\Delta Y.$

Problem 8. A 100 foot rope is hanging over a building. The linear density of the rope is $\frac{x+1}{10}$ pounds per foot (with the lighter end of the rope at the top of the building.) Find the work to bring the rope to the top. Find the work to bring half the rope to the top.

We'll use a coordinate system where the 0 is at the top of the building and 100 is at the bottom. The work to move a small segment of rope x_k feet from the top is approximately:

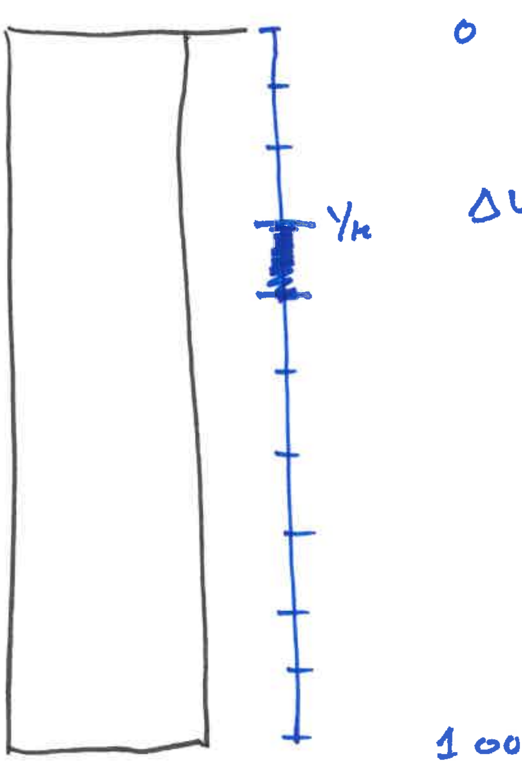
$$\Delta W_k \simeq \frac{x_k + 1}{10}(\Delta x)x_k.$$

So the total work is:

$$W = \int_{x=0}^{100} x\frac{x+1}{10}dx.$$

The work to pull half the rope is:

$$W = \int_{x=0}^{50} x\frac{x+1}{10}dx.$$

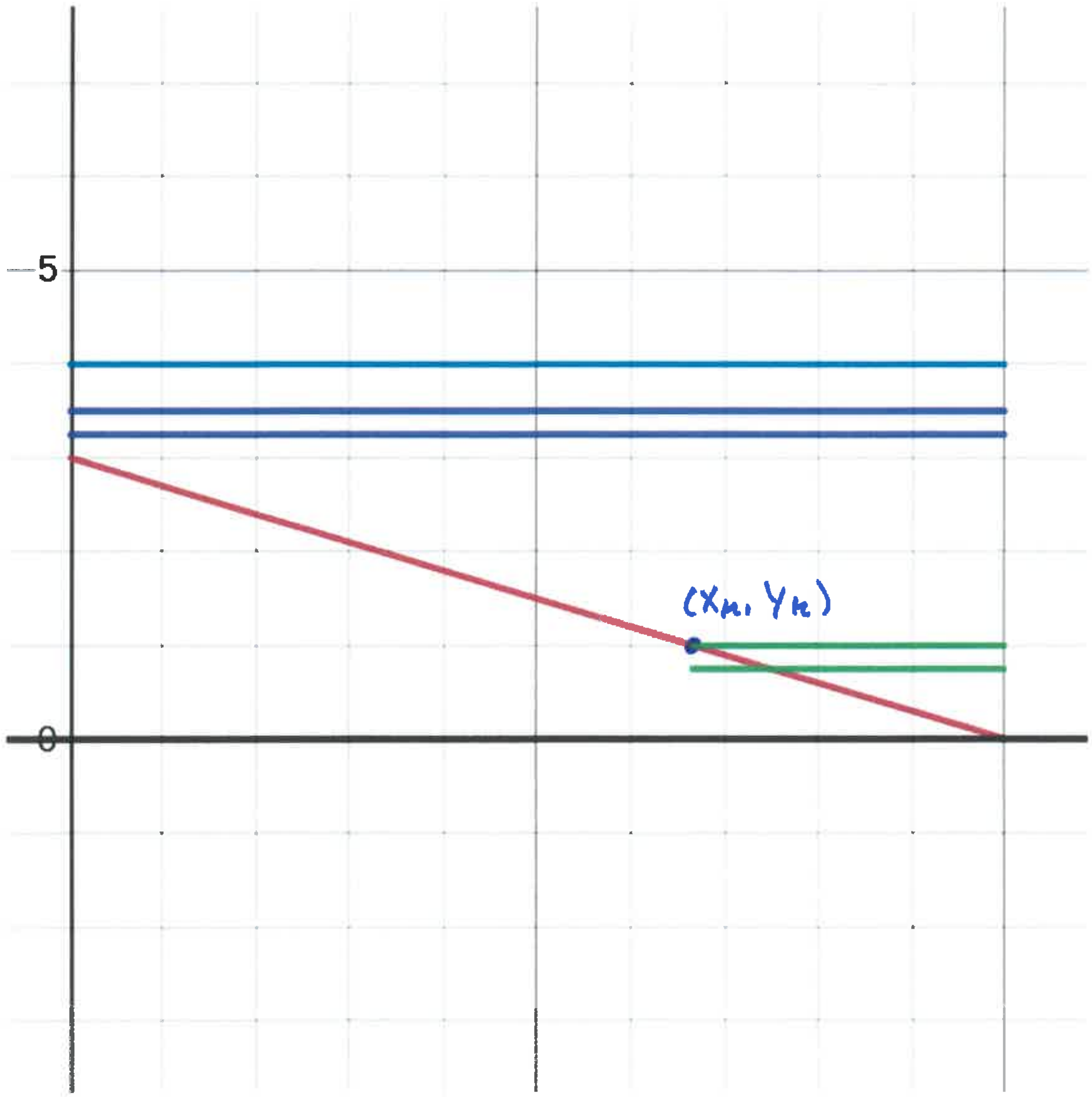


$$\Delta W_k \simeq (\text{Weight})\, y_k$$

$$\simeq \frac{y_k+1}{10}\Delta y\, y_k.$$

Continue work here.

Problem 9. A pool is 50 m long and 25 m wide. At one end the pool is 4 m deep and at the other end the pool is 1m deep. The pool slopes upward in a line from the deep to the shallow end. Find the work done in pumping the water from the pool. Find the work in pumping the water down to a level of 2 m.



The picture above is a (not to scale) drawing of a side view of the pool. The work done to move that green slab of water is approximately (below, δ is the density of water):

$$\Delta W_k \simeq \delta(25)(50 - x_k)\Delta y(4 - y_k) = \delta(25)(50 - (3 - y_k)\frac{50}{3})\Delta y(4 - y_k).$$

So, the work done to move the water out of the part of the pool from 0 to 3 meteres is:

$$W = \int_{y=0}^{3} \delta(25)(50 - (3 - y)\frac{50}{3})(4 - y)dy.$$

Now, for the other portion, note that the work done to move that purple slab is:

$$\Delta W_k \simeq \delta(25)(50)\Delta y(4 - y_k)$$

and so the work done to move this part of the water is:

$$W = \int_{y=3}^{4} \delta(25)(50)(4 - y)dy.$$

Note: the δ above is the *weight* density of water. That is, its units are N/m^3. Typically, we let ρ be the mass density and then g is acceleration due to gravity. (This is what usually appears in the solutions to previous exams, etc.) So then:

$$\delta = g\rho,$$

where:

$$g \simeq 9.8 \text{m}\,\text{s}^{-2} \qquad \rho \simeq 997 \text{kg}\,\text{m}^{-3}.$$