TEST 1 REVIEW

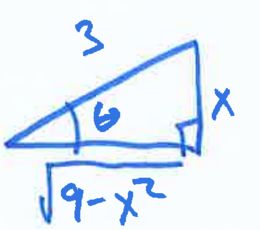
Problem 1. Compute $\int \frac{x^2}{\sqrt{9-x^2}}dx$.

Let $x = 3\sin\theta$. Then $9 - x^2 = 9 - 9\sin^2\theta$

$= 9\cos^2\theta$.

$$\int \frac{x^2}{\sqrt{9-x^2}}dx = \int \frac{9\sin^2\theta}{3\cos\theta}\, 3\cos\theta\, d\theta$$

$$= 9\int \sin^2\theta\, d\theta = \frac{9}{2}\left[\int (1-\cos 2\theta)\, d\theta\right]$$



$$= \frac{9}{2}\left[\theta - \frac{\sin 2\theta}{2}\right] = \frac{9}{2}\left[\arcsin\frac{x}{3} - \sin\theta\cos\theta\right]$$

$$= \frac{9}{2}\left[\arcsin\frac{x}{3} - \frac{x\sqrt{9-x^2}}{9}\right].$$

Problem 2. Compute $\int_{x=0}^{2} \frac{x}{\sqrt{36-4x^2}}dx$.

$$x = \frac{6}{2}\sin\theta \qquad dx = 3\cos\theta \, d\theta$$

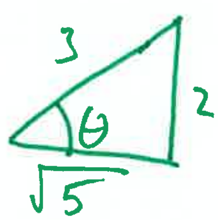
$$36 - x^2 = 36 - 36\sin^2\theta = 36\cos^2\theta$$

$$\int_0^2 \frac{x}{\sqrt{36-4x^2}}\,dx = \int_{\theta=0}^{\arcsin 2/3} \frac{3\sin\theta}{6\cos\theta}\, 3\cos\theta\, d\theta$$

$$= \frac{3}{2}\int_0^{\arcsin 2/3} \sin\theta\, d\theta$$

$$= \frac{3}{2}\left[\cos 0 - \cos\arcsin\frac{2}{3}\right]$$

$$= \frac{3}{2} - \frac{\sqrt{5}}{2}.$$



$$\cos\theta = \cos\arcsin\frac{2}{3} = \frac{\sqrt{5}}{3}$$

Problem 3. Compute $\int_{t=0}^{2} \frac{1}{\sqrt{4+t^2}} dt$.

$t = 2\tan\theta \qquad dt = 2\sec^2\theta\, d\theta$

$4+t^2 = 4+4\tan^2\theta = 4\sec^2\theta$

$$\int_0^2 \frac{1}{\sqrt{4+t^2}} dt = \int_{\theta=0}^{\pi/4} \frac{2\sec^2\theta\, d\theta}{2\sec\theta}$$

$$= \int_{\theta=0}^{\pi/4} \sec\theta\, d\theta$$

$$= \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln\left|\sec 0 + \tan 0\right|$$

$$= \ln\left|\sqrt{2} + 1\right|.$$

Problem 4. Compute $\int_{t=0}^{a} t^2\sqrt{a^2-t^2}dt$.

$t = a\sin\theta \qquad dt = a\cos\theta\, d\theta$

$a^2 - t^2 = a^2 - a^2\sin^2\theta = a^2\cos^2\theta.$

$$\int_{t=0}^{a} t^2\sqrt{a^2-t^2}\,dt = \int_{\theta=0}^{\pi/2} a^2\sin^2\theta\, a^2\cos^2\theta\, d\theta$$

$$= a^4\int_0^{\pi/2}[\cos^2\theta - \cos^4\theta]\,d\theta$$

$$= a^4\int_0^{\frac{\pi}{2}}\left(\frac{\cos 2\theta + 1}{2}\right) - \left(\frac{\cos 2\theta + 1}{2}\right)^2 d\theta$$

$$= \frac{a^4}{2}\int_0^{\frac{\pi}{2}} -\frac{\cos^2(2\theta)}{2} - \frac{1}{2}\, d\theta$$

$$= -\frac{a^4}{2}\left[\frac{\sin 4\theta}{2\cdot 8} + \frac{\theta}{2\cdot 2} - \frac{\theta}{2}\Bigg|_0^{\pi/2}\right]$$

$$= -\frac{a^4}{2}\left[\frac{-\pi/2}{4}\right] = \frac{a^4}{16}\pi.$$

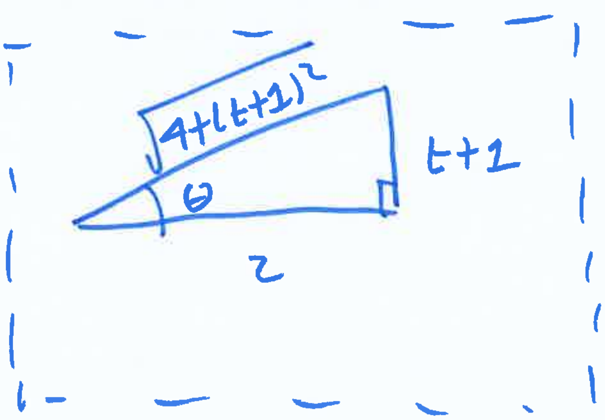
Problem 5. Compute $\int \frac{1}{t^2+2t+5}dt$.

$$t^2+2t+5 = (t+1)^2+4$$

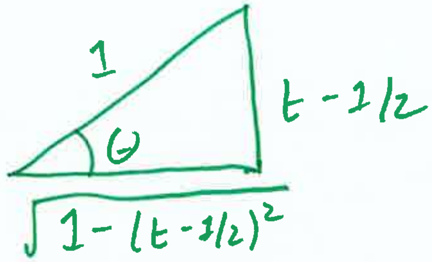
$$\int \frac{dt}{(t+1)^2+4} \qquad \text{use} \quad (t+1) = 2\tan\theta$$

$$dt = 2\sec^2\theta \, d\theta$$

$$= \int \frac{2\sec^2\theta \, d\theta}{4\sec^2\theta} = \frac{1}{2}\theta + C = \frac{\arctan\left(\frac{t+1}{2}\right)}{2} + C.$$



oops! I misread what I had. This isn't needed!



Problem 6. Compute $\int \frac{t^2}{(3+4t-4t^2)^{\frac{3}{2}}}dt.$ $= \int \frac{t^2}{\left[-4\left(t^2-t-\frac{3}{4}\right)\right]^{3/2}}$

$$\int \frac{t^2}{8\left[-(t-1/2)^2+1\right]^{3/2}}\,dt$$

Take $(t-1/2) = \sin\theta$ so $dt = \cos\theta\, d\theta$

and $1-(t-1/2)^2 = \cos^2\theta$.

$$\frac{1}{8}\int \frac{(\sin\theta+1/2)^2}{\cos^3\theta}\cos\theta\, d\theta = \frac{1}{8}\int \frac{(\sin\theta+1/2)^2}{\cos^2\theta}\,d\theta$$

$$= \frac{1}{8}\left[\int \frac{\sin^2\theta+\sin\theta+1/4}{\cos^2\theta}\,d\theta\right] = \frac{1}{8}\int \tan^2\theta + \frac{\sin\theta}{\cos^2\theta} + \frac{1}{4}\sec^2\theta\, d\theta$$

$$= \frac{1}{8}\left[\int -1 + \frac{5}{4}\sec^2\theta + \frac{\sin\theta}{\cos^2\theta}\,d\theta\right]$$

$$= \frac{1}{8}\left[-\theta + \frac{5}{4}\tan\theta + \frac{1}{\cos\theta}\right] + C = \frac{1}{8}\left[-\arcsin(t-1/2) + \frac{5}{4}\frac{t-1/2}{\sqrt{1-(t-1/2)^2}} + \frac{1}{\sqrt{1-(t-1/2)^2}}\right] + C$$

Problem 7. Compute $\int \frac{5x+1}{(2x+1)(x-1)} dx$.

$$\frac{5x+1}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1}$$

$$5x+1 = A(x-1) + B(2x+1)$$

$$x=1: \quad 6 = 3B \quad \Rightarrow \quad B=2$$

$$x=-1/2: \quad -3/2 = A(-3/2) \Rightarrow A=1$$

$$\int \frac{5x+1}{(2x+1)(x-1)} dx = \int \frac{2}{x-1} dx + \int \frac{1}{2x+1} dx$$

$$= 2\ln|x-1| + \frac{\ln|2x+1|}{2} + C.$$

Problem 8. Compute $\int \frac{x^2+x+1}{(x+1)^2(x+2)}dx$.

$$\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2}$$

repeated linear factor

So we have two terms here.

$$x^2+x+1 = A(x+1)(x+2) + B(x+2) + C(x+1)^2$$

$x=-1$: $\quad 1 = B(1) \implies B = +1$

$x=-2$: $\quad 3 = C(-2+1)^2 = C \implies C = 3$

$x=0$: $\quad 1 = 2A + 2B + C = 2A + 2 + 3 = 2A + 5$

$$\implies A = -2$$

$$\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{1}{(x+1)^2} + \frac{3}{x+2} - \frac{2}{x+1}.$$

$$\int \frac{x^2+x+1}{(x+1)^2(x+2)}dx = \int \frac{dx}{(x+1)^2} + \int \frac{3dx}{x+2} - \int \frac{2\,dx}{x+1}$$

$$= \frac{-1}{x+1} + 3\ln|x+2| - 2\ln|x+1| + C.$$

Problem 9. Compute $\int \frac{10}{(x-1)(x^2+9)}dx$.

$$\frac{10}{(x-1)\underbrace{(x^2+9)}_{\text{irreducible quadratic}}} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9}$$

so the factor goes on the bottom with linear on top.

$$10 = A(x^2+9) + (Bx+C)(x-1)$$

$x=1$: $\quad 10 = 10A \Rightarrow A = 1.$

Coeff on x^2: $\quad 0 = A + B = 1 + B \Rightarrow B = -1.$

Coff on x: $\quad 0 = -B + C = 1 + C \Rightarrow C = -1.$

$$\int \frac{10}{(x-1)(x^2+9)}dx = \int \frac{dx}{x-1} - \int \frac{x+1}{x^2+9}dx$$

$$= \ln|x-1| - \int \frac{x}{x^2+9} + \int \frac{1}{9(u^2+1)}3du$$

$x = 3u$

$$= \ln|x-1| + \frac{1}{2}\frac{1}{(x^2+9)} + \frac{1}{3}\arctan\frac{x}{3} + C.$$

Problem 10. Compute $\int \frac{x^5+x-1}{x^3+1}dx$.

First, $x=-1$ is a zero of the denominator:

$$
\begin{array}{r|l}
 & x^2 - x + 1 \\
\hline
x+1 & x^3 + 0x^2 + 0x + 1 \\
 & \underline{x^3 + x^2} \\
 & \quad -x^2 + 0x \\
 & \quad \underline{-x^2 - x} \\
 & \qquad\qquad x + 1
\end{array}
\implies x^3+1 = (x+1)\underbrace{(x^2-x+1)}_{\text{irreducible.}}
$$

Since the power on bottom is bigger, we do division (again!)

$$
\begin{array}{r|l}
 & x^2 \\
\hline
x^3 + 0x^2 + 0x + 1 & x^5 + 0x^4 + 0x^3 + 0x^2 + x - 1 \\
 & \underline{x^5 + 0x^4 + 0x^3 + x^2} \\
 & \qquad\qquad\qquad -x^2 + x - 1
\end{array}
$$

So, $$\frac{x^5+x-1}{x^3+1} = x^2 - \frac{x^2-x+1}{x^3+1} = x^2 - \frac{1}{x+1}.$$

So $$\int \frac{x^5+x-1}{x^3+1}dx = \frac{x^3}{3} - \ln|x+1| + C.$$

Problem 11. Compute $\int \frac{x^3-2x^2+2x-5}{x^4+4x^2+3}\,dx$.

First, we work with bottom:

$$\underbrace{x^4+4x^2+3}_{\substack{\text{quadratic in}\\ \text{the variable}\\ y=x^2}} = \overbrace{(x^2+1)(x^2+3)}$$

split factors

both of those are irreducible.

$$\frac{x^3-2x^2+2x-5}{(x^2+1)(x^2+3)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+3}$$

$$x^3-2x^2+2x-5 = (Ax+B)(x^2+3) + (Cx+D)(x^2+1)$$

x^3: $1 = A + C$

x^2: $-2 = B + D$

$x=0$: $-5 = 3B + D$

$\Rightarrow -3 = 2B \Rightarrow B = -3/2$

$\Rightarrow D = -1/2$

x: $2 = 3A + C$

First & Last Eq: $1 = 2A \Rightarrow A = 1/2 \Rightarrow C = 1/2$

$$\int \frac{x^3-2x^2+2x-5}{x^4+4x^2+3}\,dx = \int \frac{\frac{1}{2}x - \frac{3}{2}}{x^2+1}\,dx + \int \frac{\frac{1}{2}x - \frac{1}{2}}{x^2+3}\,dx$$

(continued)

$$= \frac{1}{2}\int \frac{x}{x^2+1}\,dx + \frac{1}{2}\int \frac{x}{x^2+3}\,dx - \frac{3}{2}\int \frac{dx}{x^2+1} - \frac{1}{2}\int \frac{dx}{x^2+3}$$

$$= \frac{1}{2}\left[\frac{1}{2}\ln(x^2+1)\right] + \frac{1}{2}\left[\ln(x^2+3)\right] - \frac{3}{2}\arctan x - \frac{1}{2}\left[\frac{1}{\sqrt{3}}\arctan\frac{x}{\sqrt{3}}\right] + C$$

Problem 12. Compute $\int \frac{\cancel{dx}}{x\sqrt{x-1}} dx$.

Let $u = \sqrt{x-1}$. Then $du = (2(x-1)^{1/2})^{-1} dx$.

$$\int \frac{dx}{x\sqrt{x-1}} = \int \frac{2\sqrt{x-1}}{x \quad u} du = \int \frac{2}{x} du$$

Here! $u^2 = x-1 \Rightarrow x = u^2+1$

$$= \int \frac{2}{u^2+1} du = 2 \arctan u + c$$

$$= 2 \arctan \sqrt{x-1} + C.$$

Problem 13. Compute $\int \frac{1}{x^2+x\sqrt{x}}dx$.

$$u = \sqrt{x} \qquad du = \frac{dx}{2\sqrt{x}} \Rightarrow dx = 2\sqrt{x}\,du$$

$$\int \frac{dx}{x^2+x\sqrt{x}} = \int \frac{2\sqrt{x}\,du}{x^2+xu} = \int \frac{2u}{u^4+u^3}\,du.$$

$$= 2\int \frac{du}{u^3+u^2} = 2\int \frac{du}{u^2(u+1)}.$$

$$\frac{1}{u^2(u+1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u+1}$$

$$1 = Au(u+1) + B(u+1) + Cu^2$$

$u=-1$: $\quad 1 = C \quad \Rightarrow \quad C = 1$

$u=0$: $\quad 1 = B \quad \Rightarrow \quad B = 1$

$u=1$: $\quad 1 = A2 + B2 + C = 2A + 4 + 1 = 2A + 5$

$\Rightarrow A = -1.$

$$2\int \frac{-1}{u} + \frac{1}{u^2} + \frac{1}{u+1}\,du = 2\left[-\ln|u| - \frac{1}{u} + \ln|u+1|\right] + C$$

$$= -2\ln\sqrt{x} + 2\ln(\sqrt{x}+1) - \frac{2}{\sqrt{x}} + C.$$

Problem 14. Compute $\int \frac{e^{2x}}{e^{2x}+3e^x+2}dx$.

Let $u = e^{x}$ so $du = e^x dx = u dx$.

$$\int \frac{e^{2x}}{e^{2x}+3e^x+2}dx = \int \frac{u^2}{u^2+3u+2}\frac{du}{u}$$

$$= \int \frac{u}{u^2+3u+2}du.$$

$$\frac{u}{u^2+3u+2} = \frac{A}{(u+1)} + \frac{B}{(u+2)}$$

$$u = A(u+2) + B(u+1)$$

$u=-1$: $-1 = A \Rightarrow A = -1$

$u=-2$: $-2 = B(-1) \Rightarrow B = 2$.

So $\int \frac{u\,du}{u^2+3u+2} = -\ln|u+1| + 2\ln|u+2| + C$.

So $\int \frac{e^{2x}}{e^{2x}+3e^x+2}dx = -\ln|e^x+1| + 2\ln|e^x+2| + C$.