SERIES

Problem 1. The series $3 - 4 + \frac{16}{3} - \frac{64}{9} + \cdots$ is geometric. Determine whether it converges or diverges. If it converges, what is its sum?

$$r = \frac{a_{n+1}}{a_n} = \frac{16/3}{-4} = -\frac{4}{3}, \quad a = 3$$

$$\sum_{k=1}^{+\infty} 3\left(-\frac{4}{3}\right)^k. \qquad \text{Since } |r| > 1, \text{ it}$$

diverges.

Problem 2. The series $\sum_{k=1}^{\infty} \frac{(-3)^{k-1}}{4^k}$ is geometric. Determine whether it converges or diverges. If it converges, what is its sum?

$$r = \frac{a_{n+1}}{a_n} = \frac{(-3)^k / 4^{k+1}}{(-3)^{k-1} / 4^k} = \frac{-3}{4}.$$

Since $|r| < 1$, it converges to:

$$a \frac{1}{1-r} = \frac{1}{4} \frac{1}{1+\frac{3}{4}}$$

$$= \frac{1}{7}.$$

Problem 3. Determine whether the series $\sum_{k=1}^{\infty} \frac{2+k}{1-2k}$ converges or diverges. If it converges, find its sum.

First, try test for divergence:

$$\lim_{k \to +\infty} \frac{2+k}{1-2k} \overset{LH}{=} \lim_{k \to +\infty} \frac{1}{-2} = -\frac{1}{2} \neq 0.$$

So, by the test for divergence, this series diverges.

Problem 4. Determine whether the series $\sum_{k=1}^{\infty} \frac{1}{4+2^{-k}}$ converges or diverges. If it converges, find its sum.

First, use the test for divergence:

$$\lim_{k} \frac{1}{4+2^{-k}} = \frac{1}{4+\lim 2^{-k}} = \frac{1}{4} \neq 0.$$

Since this is not zero, the series diverges.

Problem 5. Determine whether the series $\sum_{k=2}^{\infty} \frac{2}{k^2-1}$ converges or diverges. If it converges, find its sum.

First, try T4D: $\lim_{k} \frac{2}{k^2-1} = 0$. So this is not conclusive. Now, note that the function $f(x) = \frac{2}{x^2-1}$ is positive, continuous and decreasing so we may use integral test: $\sum_{k=1}^{+\infty} f(k)$ converges if and only if $\int_{x=1}^{+\infty} f(x)dx$ converges. Now, $k-1 \geq \frac{1}{2}k$ when $\frac{1}{2}k \geq 1$ or $k \geq 2$. So

$$\frac{1}{k^2-1} = \frac{1}{(k-1)(k+1)} \leq \frac{2}{k(k+1)} \leq \frac{2}{k^2}.$$ So

$$\lim_{b\to\infty} \int_2^{+\infty} \frac{dx}{x^2-1} \leq 2\lim_b \int_2^b \frac{dx}{x^2} = 2\lim_b \left[\frac{1}{2} - \frac{1}{b}\right] = 1.$$

Since the integral converges the series does, too.

Problem 6. Determine whether the series $\sum_{k=1}^{\infty} \frac{3}{k(k+3)}$ converges or diverges. If it converges, find its sum.

Using TAD, $\lim_{k} \frac{3}{k(k+3)} = 0$ so TAD is not conclusive. So we use IT:

$$\lim_{b} \int_1^b \frac{3}{x(x+3)}\,dx = \lim_{b} \int_1^b \frac{3\,dx}{x^2} = 3\lim_{b}\left[1 - \frac{1}{b}\right] = 3.$$

So it converges. To compute the sum:

$$\frac{3}{k(k+3)} = \frac{A}{k} + \frac{B}{k+3} \Rightarrow 3 = A(k+3) + Bk.$$

For $k=0$: $3 = 3A$; $k=-3$: $3 = -3B$. So

$$\frac{3}{k(k+3)} = \frac{1}{k} - \frac{1}{k+3}.$$

$$\sum_{k=1}^{N} \frac{3}{k(k+3)} = \left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{N-3} + \frac{1}{N-2} + \frac{1}{N-1} + \frac{1}{N}\right] -$$

$$\left[\frac{1}{4} + \frac{1}{5} + \dots + \frac{1}{N} + \frac{1}{N+1} + \frac{1}{N+2} + \frac{1}{N+3}\right]$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{N+1} + \frac{1}{N+2} + \frac{1}{N+3} \to \frac{6+3+2}{6} = \frac{11}{6}.$$

Problem 7. Find the values of x for which the series converges and find the sum: $\sum_{k=1}^{\infty}(-5)^k x^k$.

This is a geometric series. We need it to be in the form $\sum_{k=1}^{+\infty} a r^{k-1}$. So we write it as $\sum_{k=1}^{+\infty} (-5x)^k = \sum_{k=1}^{+\infty} (-5x)(-5x)^{k-1}$.

We see $|r| = |5x|$ so it converges if and only if $|x| < 1/5$. In this case the sum is:

$$(-5x)\frac{1}{1+5x} = \frac{-5x}{1+5x}.$$

Problem 8. Find the values of x for which the series converges and find the sum: $\sum_{k=1}^{\infty} \frac{(x-2)^k}{3^k}$.

Again, this is a geometric series. We put it in standard form $\sum_{k=1}^{\infty} \frac{x-2}{3}\left(\frac{x-2}{3}\right)^{k-1}$.

This converges if and only if $\left|\frac{x-2}{3}\right| < 1$.

That is $|x-2| < 3$. Or $-1 < x < 5$.

For these values of x the sum is:

$$\frac{x-2}{3}\,\frac{1}{1-\frac{x-2}{3}} = \frac{x-2}{3-(x-2)} = \frac{x-2}{5-x}.$$

Problem 9. Find the values of x for which the series converges and find the sum: $\sum_{k=0}^{\infty} \frac{2^k}{x^k}$.

This is also geometric. But we need to start the sum at index 1. So, let $l = k+1$. Then this is (note $k = l-1$)

$$\sum_{l=1}^{+\infty} \left(\frac{2}{x}\right)^{l-1}.$$

This converges if and only if $\left|\frac{2}{x}\right| < 1$. That is $|x| > 2$. For these values of x the sum is:

$$\frac{1}{1-\frac{2}{x}} = \frac{1}{\frac{1}{x}(x-2)} = \frac{x}{x-2}.$$

Problem 10. Use the integral test to determine whether $\sum_{k=1}^{\infty} k^{-3}$ is convergent or divergent.

Then $f(k) = \frac{1}{k^3}$ and this is continuous, decreasing, positive so we can apply integral test:

$$\int_1^{+\infty} \frac{dx}{x^3} = \lim_{b} \int_1^b \frac{dx}{x^3}$$

$$= \frac{1}{2} \lim_{b} \left[\frac{1}{1^2} - \frac{1}{b^2} \right]$$

$$= \frac{1}{2}.$$

Since this converges, the series converges

Problem 11. Use the integral test to determine whether $\sum_{k=1}^{\infty} \frac{2}{5k-1}$ is convergent or divergent.

The function $f(x) = \frac{2}{5k-1}$ is continuous, decreasing, positive so we may apply integral test. Also, note $\frac{2}{5x-1} > \frac{2}{5x}$. Now:

$$\int_1^{+\infty} \frac{2\,dx}{5x-1} = \lim_{b} \int_1^b \frac{2}{5x-1}\,dx$$

$$> \lim_{b} \int_1^b \frac{2}{5x}\,dx$$

$$= \frac{2}{5} \lim_{b} \left[\ln b \right]$$

$$= +\infty.$$

Since the integral diverges, so does the series.

Problem 12. Use the integral test to determine whether $\sum_{k=1}^{\infty} k^2 e^{-k^3}$ is convergent or divergent.

Note if $f(x) = x^2 e^{-x^3}$ then:

$$f'(x) = 2x e^{-x^3} + (-3x^2) x^2 e^{-x^3}.$$

Then $2xe^{-x^3} - 3x^2 x^2 e^{-x^3} < 0$ only if

$2 < 3x^3$ which is true if

and only if $x^3 > \frac{2}{3}$ which is true if

$x > 1$. So, we determine convergence or

divergence of $\sum_{k=1}^{\infty} k^2 e^{-k^3}$. So

$$\int_1^b x^2 e^{-x^3}\, dx = \int_1^{b^3} e^{-u} \frac{du}{3} = \frac{1}{3}\left[e^{-1} - e^{-b^3}\right].$$

$u = x^3$
$du = 3x^2\, dx$

Now, as $b \to \infty$, this goes to $\frac{1}{3e}$. So

the integral converges, so does the series.

Problem 13. Use the integral test to determine whether $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ is convergent or divergent.

Since $x \ln x$ increases, $\frac{1}{x \ln x}$ decreases and is continuous and positive. So we may use integral test.

$$\int_{x=2}^{+\infty} \frac{dx}{x \ln x} = \lim_{b} \int_{2}^{b} \frac{dx}{x \ln x}$$

$u = \ln x \quad du = \frac{dx}{x}$

$$= \lim_{b} \int_{\ln 2}^{\ln b} \frac{du}{u}$$

$$= \lim_{b} \left[\ln \ln b - \ln \ln 2 \right]$$

$$= +\infty.$$

Problem 14. Use the integral test to determine whether $\sum_{k=1}^{\infty} ke^{-k}$ is convergent or divergent.

To do integral test, we need to make sure xe^{-x} is (eventually) decreasing. ~~Note~~ that $f'(x) = e^{-x} - xe^{-x} < 0$ when

$1 - x < 0$ or $x > 1$. Then

$$\int_1^{+\infty} xe^{-x}\,dx = \lim_{b} \int_1^b xe^{-x}\,dx$$

$$= \lim_{b} \left[-xe^{-x}\Big|_1^b + \int_1^b e^{-x}\,dx \right]$$

$$= \lim_{b} \left[e^{-1} - be^{-b} + e^{-1} - e^{-b} \right]$$

$$= 2e^{-1}.$$

Since the integral converges, so does the series.