



Instructor: Dr. Rosanna Pearlstein

Math 251 – Spring 2022  
“Week-in-Review”

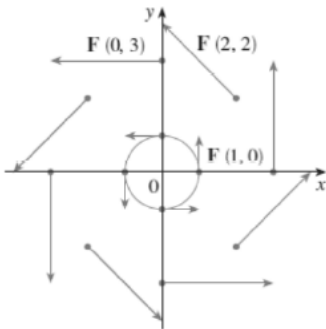
**Wir 10: Sections 16.1, 16.2, 16.3, 16.4, 16.5**

**Section 16.1**

Definition: A vector field in two dimension is a function  $\mathbf{F}$  that assigns to each point  $(x, y)$  in  $D \subset \mathbb{R}^2$  a two dimensional vector,  $\mathbf{F}(x, y)$ .

In two dimension, the vector field lies entirely in the  $xy$  plane.

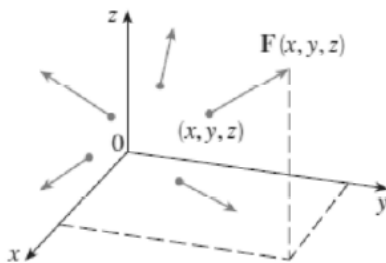
Here is a vector field in  $\mathbb{R}^2$ :



Definition: A vector field in three dimension is a function  $\mathbf{F}$  that assigns to each point  $(x, y, z)$  in  $D \subset \mathbb{R}^3$  a three dimensional vector,  $\mathbf{F}(x, y, z)$ .

In three dimension, the vector field is in space.

Here is a vector field in  $\mathbb{R}^3$ :



*With thanks to Amy Austin for generously sharing her WIR problems.*

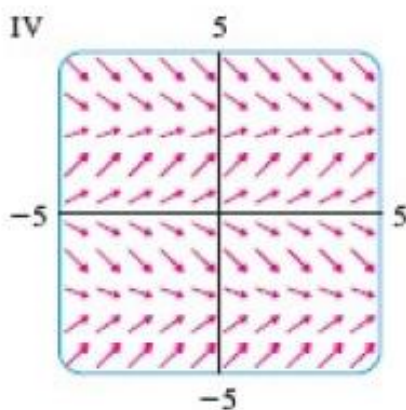
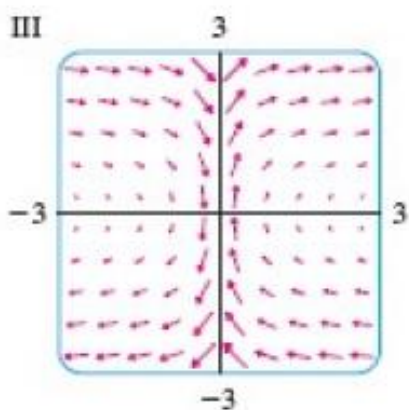
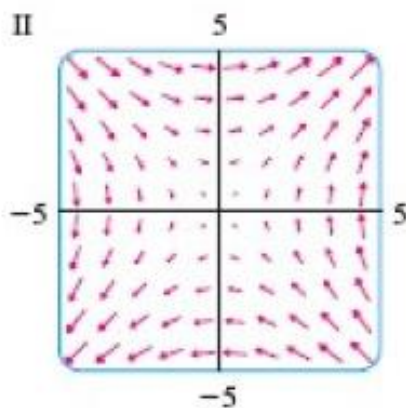
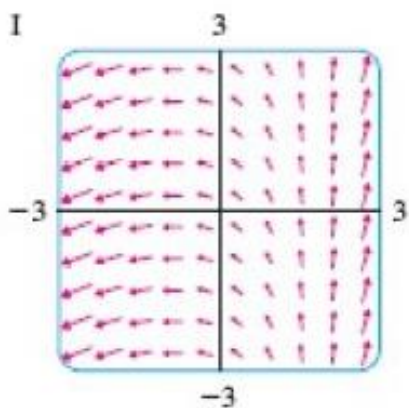
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In order to match  $\mathbf{F}$  with its vector field, choose a several points,  $(x, y)$ , in each quadrant, and look at the *direction* of  $\mathbf{F}(x, y)$ . To narrow down further, look at the behavior of the components. Often times, it is a process of elimination.

Problem 1. Match each vector field equation with its graph:

- a)  $\mathbf{F}(x, y) = \langle y, x \rangle$
- b)  $\mathbf{F}(x, y) = \langle 1, \sin y \rangle$
- c)  $\mathbf{F}(x, y) = \langle x - 2, x + 1 \rangle$
- d)  $\mathbf{F}(x, y) = \langle y, \frac{1}{x} \rangle$



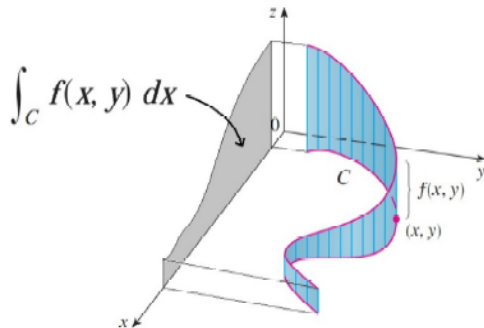
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Section 16.2

**Definition:** If  $f$  is defined on a smooth curve  $C$  defined as  $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ ,  $a \leq t \leq b$ , then the line integral of  $f$  along  $C$  is

$$\int_C f(x, y) ds = \int_a^b (f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_a^b (f(x(t), y(t)) |\mathbf{r}'(t)| dt$$



In order to find a line integral along a curve  $C$ , we must first parameterize the curve. Sometimes, the parameterization will be given explicitly, other times you must parameterize the curve.

**Problem 2.** Evaluate  $\int_C (2x + y) ds$ , where  $C$  is defined as  $\mathbf{r}(t) = \langle 2 + t, 3 - t \rangle$ ,  $0 \leq t \leq 1$ .

**Problem 3.** Set up but do not evaluate  $\int_C (2x + x^2 y) ds$ , where  $C$  is the arc of the curve  $y = x^2$  from  $(1, 1)$  to  $(2, 4)$  using two different parameterizations.

**Problem 4.** Evaluate  $\int_C (x^2 + y) ds$  where  $C$  consists of the line segment from the point  $(1, 4)$  to  $(3, -1)$ .

**Problem 5.** Evaluate  $\int_C (x + y) ds$ , where  $C$  is the top half of the circle  $x^2 + y^2 = 4$ , oriented counterclockwise.

**Problem 6.** Set up but do not evaluate  $\int_C (2 + x^2 y) ds$ , where  $C$  is the arc of the curve  $x = y^2$  from  $(1, -1)$  to  $(4, 2)$  and then along the line segment from the point  $(4, 2)$  to the point  $(3, 7)$ .



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**Line Integrals over vector fields:** Suppose now we are moving a particle along a curve  $C$  through a vector (force) field,  $\mathbf{F}$ . We define the **line integral of  $\mathbf{F}$  along  $C$**  to be

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b (\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)) dt$$

**Problem 7.** Find  $\int_C \mathbf{F} \cdot \mathbf{r}$ , where  $C$  is defined by  $\mathbf{r}(t) = \langle t, t^2, t^4 \rangle$ ,  $0 \leq t \leq 1$ , and  $\mathbf{F}(x, y, z) = \langle x, z^2, -4y \rangle$ .

**Problem 8.** Find the work done by the force field  $\mathbf{F}(x, y) = \langle x^2, xy \rangle$  in moving an object counterclockwise around the right half of the circle  $x^2 + y^2 = 9$ .

**Definition:** Let  $C$  be a smooth curve defined by the parametric equations  $x = x(t)$ ,  $y = y(t)$  for

$a \leq t \leq b$ . The **line integral of  $f$  along  $C$  with respect to  $x$**  is  $\int_C f(x, y) dx = \int_a^b (f(x(t), y(t)) x'(t)) dt$ .

The **line integral of  $f$  along  $C$  with respect to  $y$**  is  $\int_C f(x, y) dy = \int_a^b (f(x(t), y(t)) y'(t)) dt$

**Problem 9.**  $\int_C y dx + x^2 dy$ , where  $C$  is described by  $\mathbf{r}(t) = \langle 3e^t, e^{2t} \rangle$ ,  $0 \leq t \leq 1$ .

**Problem 10.** Evaluate  $\int_C x dx + y dy$ , where  $C$  is the arc of the parabola  $x = 4 - y^2$  from  $(-5, -3)$  to  $(3, 1)$ .

**Problem 11.** Evaluate  $\int_C (x + y) dz + (y - x) dy + z dx$  where  $C$  is described by  $x = t^4$ ,  $y = t^3$ ,  $z = t^2$ ,  $0 \leq t \leq 1$ .



### Section 16.3

In section 16.2, we learned how to find a line integral over a vector field  $\mathbf{F}$  along a curve  $C$  that is parametrized by  $\mathbf{r}(t)$ ,  $a \leq t \leq b$ .

**Problem 1.** Suppose we are moving a particle from the point  $(0, 0)$  to the point  $(2, 4)$  in a force field  $\mathbf{F}(x, y) = \langle y^2, x \rangle$ . Find  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where

- a.) The particle travels along the line segment from  $(0, 0)$  to  $(2, 4)$ .
- b.) The particle travels along the curve  $y = x^2$  from  $(0, 0)$  to  $(2, 4)$ .

Note: Although the end points are the same, the value of the line integral is **different** because the **paths** are different. In this section, we will learn under what conditions the line integral is independent of the path taken.

**Definition:** If  $\mathbf{F}$  is a continuous vector field, we say that  $\int_C \mathbf{F} \cdot d\mathbf{r}$  is **independent of path** if and only if  $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$  for any two paths  $C_1$  and  $C_2$  with the same starting and ending points. In other words, the line integral is the same **no matter what path** you travel on as long as the endpoints are the same.

**Definition:** A vector field  $\mathbf{F}$  is called a **conservative vector field** if it is the gradient of some scalar function  $f$ , that is there exists a function  $f$  so that  $\mathbf{F} = \nabla f$ . We call  $f$  the **potential function**.

**Problem 2.** Consider  $f(x, y) = x^2y - y^3$ . Find the gradient and explain why it is conservative. What is the potential function?



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Recall the Fundamental Theorem of Calculus tells us that  $\int_a^b f'(x)dx = f(b) - f(a)$ .

Since  $\nabla f = \langle f_x, f_y \rangle$ , we can think of the potential function,  $f$ , as some sort of antiderivative of  $\nabla f$ . Hence  $\int \mathbf{F} \cdot d\mathbf{r} = \int \nabla f \cdot d\mathbf{r}$ .

**Fundamental Theorem for Line Integrals:** Let  $C$  be a smooth curve given by the vector function  $\mathbf{r}(t)$ ,  $a \leq t \leq b$ . Let  $\mathbf{F}$  be a conservative vector field. Let  $f$  be a differentiable function of two or three variables whose gradient vector,  $\nabla f$ , is continuous on  $C$ . Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Note: **Line integrals of conservative vectors fields are independent of path** because in a conservative vector field, the line integral is computed by only using the **endpoints** of the domain! Therefore, if we are in a conservative vector field, the line integral along a curve  $C$  in that vector field will be the same **no matter what curve we travel across** that connects the endpoints together. **WHICH MEANS WE DON'T EVEN NEED TO PARAMETERIZE THE CURVE!**

Question: How do we determine if a vector field is conservative, and if so, how do we find the potential function? The 'test for conservative' we use depends on whether  $\mathbf{F}$  is in  $\mathbb{R}^2$  or  $\mathbb{R}^3$ .

**Theorem:**  $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle = P\mathbf{i} + Q\mathbf{j}$  is a conservative vector field, where  $P$  and  $Q$  have continuous first-order partial derivatives on a domain  $D$ , if and only if  $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ .

Note: This above criteria to determine if a vector field is conservative works only for  $\mathbb{R}^2$ .

**Problem 3.** Is  $\mathbf{F}(x, y) = \langle 3x^2 - 4y, 4y^2 - 2x \rangle$  a conservative vector field? If so, find a function  $f$  so that  $\mathbf{F} = \nabla f$ .

**Problem 4.** Is  $\mathbf{F}(x, y) = \langle x + y, x - 2 \rangle$  a conservative vector field? If so, find a function  $f$  so that  $\mathbf{F} = \nabla f$ .



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**Problem 5.** Given  $\mathbf{F}(x, y) = \langle 2xy^3, 3x^2y^2 \rangle$ . Evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$  where  $C$  is the curve given by  $\mathbf{r}(t) = \langle t^3 + 2t^2 - t, 3t^4 - t^2 \rangle$ ,  $0 \leq t \leq 2$ .

**Problem 6.** Let  $\mathbf{F}(x, y) = \langle 3 + 2xy^2, 2x^2y \rangle$ . Evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$  where  $C$  is the arc of the hyperbola  $y = \frac{1}{x}$  from  $(1, 1)$  to  $\left(4, \frac{1}{4}\right)$ .

**Problem 7.** Given  $\mathbf{F}(x, y) = \langle 3x^2 - 4y, 4y^2 - 2x \rangle$ . Evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$  where  $C$  is the curve given by  $\mathbf{r}(t) = \langle t^2, t^2 + t - 2 \rangle$ ,  $0 \leq t \leq 1$ .

## Section 16.4

**Green’s Theorem:** Let  $C$  be a positively oriented (counterclockwise) piecewise-smooth simple closed curve in the plane and let  $D$  be the region bounded by  $C$ . If  $P$  and  $Q$  have continuous partial derivatives on an open region that contains  $D$ , then

$$\oint_C Pdx + Qdy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

This says that the line integral over a simple closed curve  $C$  is equal to a double integral over the area of the region  $D$  the curve  $C$  encloses.

Note: We only use Green’s theorem if we are on a **positively oriented closed curve**. If the curve is not positively oriented, then change the sign of the line integral. If not explicitly stated, assume counterclockwise orientation.

**Problem 8.** Evaluate  $\oint_C y^2 dx + x dy$  where  $C$  is the triangular path from  $(1, 1)$  to  $(3, 1)$  to  $(2, 2)$  then back to  $(1, 1)$ .

**Problem 9.** Evaluate  $\oint_C y^2 dx + x^2 dy$  where  $C$  is the boundary of the region bounded by the semicircle  $y = \sqrt{4 - x^2}$  and the  $x$  axis. Assume positive orientation.

**Problem 10.** Suppose a particle travels one revolution clockwise around the unit circle under the force field  $\mathbf{F}(x, y) = \langle e^x - y^3, \cos(y) + x^3 \rangle$ . Find the work done.

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Section 16.5

**Definition:** The del operator, denoted by  $\nabla$ , is defined as  $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

**Definition of curl and divergence:**

**Problem 11.** Find the divergence and curl of  $\mathbf{F} = \langle xy, xz, xyz^2 \rangle$ .

**Theorem:** If  $\mathbf{F}$  is a vector field defined on all of  $\mathfrak{R}^3$  whose component functions have continuous partial derivatives and  $\text{curl } \mathbf{F} = \mathbf{0}$ , then  $\mathbf{F}$  is a conservative vector field. This gives us a way to determine whether a vector function on  $\mathfrak{R}^3$  is conservative.

**Problem 12.** If  $\mathbf{F} = \langle x, e^y \sin z, e^y \cos z \rangle$ , Find  $\int_C \mathbf{F} \cdot d\mathbf{r}$  where  $\mathbf{r}(t) = \langle t^4, t, 2t^2 \rangle$ , for  $1 \leq t \leq 2$ .

**Problem 13.** If time permits, discuss what operators make sense (similar to weassign section 16.5 problem 8)