Week in Review Session ♯10 (Sections 5.1-5.3)

This document contains the answers and video solutions to the posed problems. Click the red box to watch the video solution. You can also watch all videos by viewing the Session 10 playlist. In the event that this weekly review handout is updated, the Session video playlist will reflect the most updated problem set. Closed captions are available for all videos and the speed of the videos may be adjusted inside of "Settings" or the cog in the bottom right corner.

1. Determine the radius of convergence for the following power series:

   (a) $\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$

   Answer: The series converges for all $x \in \mathbb{R}$, so the series has an infinite radius of convergence.

   Click here to see video solution to problem #1(a)

   (b) $\sum_{n=1}^{\infty} \frac{(-1)^n n^2 (x+2)^n}{3^n}$

   Answer: The radius of convergence $R = 3$

   Click here to see video solution to problem #1(b)

2. For the equation $(x^2+1)y'' + xy' - y = 0$

   (a) Determine a lower bound for the radius of convergence of the series solutions of the differential equation about $x_0 = 0$.

   Answer: The lower bound is 1

   Click here to see video solution to problem #2(a)

   (b) Seek its power series solution about $x_0 = 0$; find the recurrence relation.

   Answer: $\sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$. The recurrence relation is $(n+2)a_{n+2} + (n-1)a_n = 0$ for all $n = 0, 1, 2, ..$

   Click here to see video solution to problem #2(b)

(c) Find the general term of each solution $y_1(x)$ and $y_2(x)$.

Answer: $y_1 = 1 + \sum_{k=1}^{\infty} (-1)^{k-1} \dfrac{(2k-3)!!}{(2k)!!} x^{2k}$ where $(-1)!! = 1$, $\quad y_2 = x$

Click here to see video solution to problem #2(c)

(d) Find the first four terms in each of two solutions y_1 and y_2. Show that $W[y_1, y_2](0) \neq 0$.

Answer: $y_1 = 1 + \dfrac{1}{2}x^2 - \dfrac{1}{(2)(4)}x^4 + \dfrac{3(1)}{(2)(4)(6)}x^6 - \ldots$ and $\quad y_2 = x$

$$W[y_1, y_2](0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$
$$= 1$$

Click here to see video solution to problem #2(d)

3. For the equation $(x^2 + 1)y'' - 6y = 0$

(a) Determine a lower bound for the radius of convergence of the series solutions of the differential equation about $x_0 = 0$.

Answer: The lower bound is 1

Click here to see video solution to problem #3(a)

(b) Seek its power series solution about $x_0 = 0$; find the recurrence relation.

Answer: $\sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=0}^{\infty} 6a_n x^n = 0$. The recurrence relation is $a_{n+2} = -\dfrac{n-3}{n+1} a_n$ for all $n = 0, 1, 2, \ldots$

Click here to see video solution to problem #3(b)

(c) Find the general term of each solution $y_1(x)$ and $y_2(x)$.

Answer: $y_1 = 1 + 3x^2 + x^4 + \sum_{k=3}^{\infty} \frac{3(-1)^k}{(2k-1)(2k-3)} x^{2k}$, $\qquad y_2 = x + x^3$

Click here to see video solution to problem #3(c)

Video errata: At the 41:18 mark, we should have that $\sum_{k=0}^{\infty} a_{2k+1} x^{2k+1} = a_1(x + x^3)$. At the 44:01 mark, we should have that $\sum_{k=0}^{\infty} a_{2k} x^{2k} = a_0 \left(1 + 3x^2 + x^4 + \sum_{k=3}^{\infty} \frac{3(-1)^k}{(2k-1)(2k-3)} x^{2k}\right)$. This has been digitally added in the video.

(d) Find the first four terms in each of two solutions y_1 and y_2. Show that $W[y_1, y_2](0) \neq 0$.

Answer: $y_1 = 1 + 3x^2 + x^4 - \frac{1}{5}x^6 + \ldots \qquad y_2 = x + x^3$

$$W[y_1, y_2](0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Click here to see video solution to problem #3(d)

4. Seek a power series solution for the initial value problem

$$y'' - (1+x)y = 0, \quad y(0) = 1, y'(0) = -2$$

up to the terms of degree 5. Then do the same for finding the general solution of the equation.

Answer: $y = 1 - 2x + \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{8}x^4 + \frac{1}{60}x^5 + ...$ For the general solution of the equation, we have that $y = a_0y_1 + a_1y_2$, where $y_1 = 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{30}x^5 + ...$ and $y_2 = x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5 + ...$

Click here to see video solution to problem #4

Video errata: At the 9:07 mark, we should have that $12a_4 = -\frac{3}{2}$ and so $a_4 = -\frac{1}{8}$. At the 10:11 mark, we should have that $20a_5 = \frac{1}{3}$ and so $a_5 = \frac{1}{60}$. This is corrected at the 10:52 mark.

5. For the following equation, determine $\phi''(x_0)$ and $\phi'''(x_0)$, for the given point x_0 if $y = \phi(x)$ is a solution of the given initial-value problem.

$$y'' + x^2y' + (\sin x)y = 0; \quad y(0) = a_0, \quad y'(0) = a_1$$

Answer: We have that $\phi''(0) = 0$, $\phi'''(0) = -a_0$, and $\phi(x) = a_0(1 - \frac{x^3}{6} + ...) + a_1(x + ...)$

Click here to see video solution to problem #5