Week in Review Session ♯11 (Sections 7.1-7.9)

This document contains the answers and video solutions to the posed problems. Click the red box to watch the video solution. You can also watch all videos by viewing the Session 11 playlist. In the event that this weekly review handout is updated, the Session video playlist will reflect the most updated problem set. Closed captions are available for all videos and the speed of the videos may be adjusted inside of "Settings" or the cog in the bottom right corner.

1. Transform the given initial value problem into an initial value problem for three first-order equations, and write it in matrix form.

$$u''' + p(t)u'' + q(t)u' + r(t)u = g(t), \quad u(0) = u_0, \quad u'(0) = u_1, \quad u''(0) = u_2$$

Answer: $\overrightarrow{X}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -r(t) & -q(t) & -p(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ g(t) \end{pmatrix}, \quad \overrightarrow{X}(0) = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix}$

Click here to see video solution to problem #1

2. Given the two vectors

$$\mathbf{x_1(t)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mathbf{e^t} \text{ and } \mathbf{x_2(t)} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \mathbf{e^{2t}},$$

find a differential equation $\mathbf{x'} = A\mathbf{x}$ for which $\mathbf{x_1(t)}$ and $\mathbf{x_2(t)}$ are solutions.

Answer: $\mathbf{x'} = \begin{pmatrix} 5 & -2 \\ 6 & -2 \end{pmatrix} \mathbf{x}$

Click here to see video solution to problem #2

3. Find the general solution of the system and the fundamental matrix. Classify the type of the critical point (0,0), and determine whether it is stable or unstable. Sketch the phase portrait.

(a) $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x}$

Answer: The general solution is $\vec{x}(t) = C_1 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, the fundamental matrix is $\begin{pmatrix} e^{-t} & 2e^{2t} \\ 2e^{-t} & e^{2t} \end{pmatrix}$, and the critical point $(0,0)$ is a saddle point and unstable. Refer to video for sketch of phase portrait.

Click here to see video solution to problem #3(a)

(b) $\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$

Answer: The general solution is $\vec{x}(t) = C_1 e^{t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{t} \left[t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]$, the fundamental matrix is $\begin{pmatrix} 2e^{t} & (2t+1)e^{t} \\ e^{t} & te^{t} \end{pmatrix}$, and the critical point $(0,0)$ is a improper node and unstable. Refer to video for sketch of phase portrait.

Click here to see video solution to problem #3(b)

(c) $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \mathbf{x}$

Answer: The general solution is

$$\vec{x}(t) = C_1 e^{-t} \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(t) \right] + C_2 e^{-t} \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} \sin(t) + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos(t) \right],$$

the fundamental matrix is $\begin{pmatrix} e^{-t}\cos(t) & e^{-t}\sin(t) \\ e^{-t}(2\cos(t) + \sin(t)) & e^{-t}(2\sin(t) - \cos(t)) \end{pmatrix}$, and the critical point $(0,0)$ type is a spiral source and asymptotically stable. Refer to video for sketch of phase portrait.

Click here to see video solution to problem #3(c)

4. Classify the type and stability of the equilibrium point(s) of the system

$$\mathbf{x}' = \begin{pmatrix} \alpha - 1 & \alpha + 1 \\ -2/3 & 1/3 \end{pmatrix} \mathbf{x}$$

for different values of the parameter α.

Answer:

- If $\alpha < \alpha_1 = -\dfrac{1}{3} \implies (0,0)$ is a saddle point and is unstable.
- If $\alpha_1 = -\dfrac{1}{3} < \alpha < \alpha_2 = \dfrac{8}{3} - 2\sqrt{2} \implies (0,0)$ is an asymptotically stable sink (nodal sink).
- If $\alpha_2 = \dfrac{8}{3} - 2\sqrt{2} < \alpha < \alpha_3 = \dfrac{2}{3} \implies (0,0)$ is an asymptotically stable spiral sink.
- If $\alpha_3 = \dfrac{2}{3} < \alpha < \alpha_4 = \dfrac{8}{3} + 2\sqrt{2} \implies (0,0)$ is an unstable spiral source.
- If $\alpha > \alpha_4 = \dfrac{8}{3} + 2\sqrt{2} \implies (0,0)$ is an unstable source (nodal source).
- If $\alpha = \alpha_1$, we have a line of equilibria which is asymptotically stable.
- If $\alpha = \alpha_2 \implies$ (0,0) is an asymptotically stable improper node.
- If $\alpha = \alpha_3 \implies$ (0,0) is a stable center.
- If $\alpha = \alpha_4 \implies$ (0,0) is unstable improper node.

Click here to see video solution to problem #4

5. Find the general solution of the nonhomogeneous system.

$$\mathbf{x}' = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} t^{-1} \\ 2t^{-1} + 4 \end{pmatrix}, \qquad t > 0$$

Answer: $\vec{x}(t) = (c_1 + \ln(t) + \frac{8}{5}t) \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (c_2 e^{-5t} + \frac{4}{25}) \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Click here to see video solution to problem #5