Week in Review Session ♯12 Final Exam Review

This document contains the answers and video solutions to the posed problems. Click the red box to watch the video solution. You can also watch all videos by viewing the Session 12 playlist. In the event that this weekly review handout is updated, the Session video playlist will reflect the most updated problem set. Closed captions are available for all videos and the speed of the videos may be adjusted inside of "Settings" or the cog in the bottom right corner.

Review for Final Exam (Please note that this review may not cover all sections of your Final Exam)

1. (Chapter 2) Find the general solution of the given differential equation.
   (a) $2\sqrt{x}\,y' = \sqrt{1-y^2}$

   Answer: $y = \sin(\sqrt{x} + C)$, equilibrium solutions at $y = \pm 1$

   Click here to see video solution to problem #1(a)

   (b) $ty' + y = 3t\cos t, \qquad t > 0.$

   Answer: $y = \dfrac{3(t\sin(t) + \cos(t)) + C}{t}$

   Click here to see video solution to problem #1(b)

2. (Chapter 2) Given the differential equation

$$y'(t) = y^3 - 2y^2 + y$$

   (a) Find the equilibrium solutions.
   (b) Graph the phase line. Classify each equilibrium solution as either stable, unstable, or semistable.
   (c) Sketch the graph of some solutions.
   (d) If $y(t)$ is the solution of the equation satisfying the initial condition $y(0) = y_0$, where $-\infty < y_0 < \infty$, determine the behavior of $y(t)$ as t increases.
   (e) Do any solutions 'blow up in finite time,' namely, do they admit a vertical asymptote?

   Answer: For parts (a-e), see videos below.

   Click here to see video solution to problem #2(a) and #2(b)

   Click here to see video solution to problem #2(c),#2(d), and #2(e)

3. (Chapter 2) Solve $y' = \dfrac{(1+x)e^x}{xe^x - ye^y}$.

   Answer: $\dfrac{y^2}{2} + xe^x e^{-y} = C$

   Click here to see video solution to problem #3

4. (Chapter 3) Find the solution to the given initial value problem.

   $y'' - 4y' + 3y = 0, \quad y(0) = 3, \quad y'(0) = 4.$

   Answer: $y = \dfrac{5}{2}e^t + \dfrac{1}{2}e^{3t}$

   Click here to see video solution to problem #4

5. (Chapter 3) If the Wronskian of f and g is $t\cos(t) - \sin(t)$, and if $u = 2f - 3g$, and $v = f + g$, find the Wronskian of u and v.

   Answer: $W[u, v] = 5W[f, g] = 5(t\cos(t) - \sin(t))$

   Click here to see video solution to problem #5

6. (Chapter 3) Find the general solution of the equation $y'' + 2y' + y = 4e^{-t}$.

   Answer: $y(t) = e^{-t}(c_1 + c_2 t + 2t^2)$

   Click here to see video solution to problem #6

7. (Chapter 3) Solve the differential equation

   $$t^2 y'' + 2ty' - 2y = t, \quad t > 0$$

   assuming $y_1(t) = t$ is a solution to the corresponding homogeneous equation.

   Answer: $y(t) = c_1 t + c_2 t^{-2} + \dfrac{1}{3} t\ln(t)$

   Click here to see video solution to problem #7

8. (Chapter 5) For the equation $(x^2+1)y'' - 6y = 0$

(a) Determine a lower bound for the radius of convergence of the series solutions of the differential equation about $x_0 = 0$.

Answer: The lower bound is 1

Click here to see video solution to problem #8(a)

(b) Seek its power series solution about $x_0 = 0$; find the recurrence relation.

Answer: $\sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} 6a_n x^n = 0$. The recurrence relation is $a_{n+2} = -\frac{n-3}{n+1}a_n$ for all $n = 0, 1, 2, ...$

Click here to see video solution to problem #8(b)

(c) Find the general term of each solution $y_1(x)$ and $y_2(x)$.

Answer: $y_1 = 1 + 3x^2 + x^4 + \sum_{k=3}^{\infty} \frac{3(-1)^k}{(2k-1)(2k-3)}x^{2k}$, $\quad y_2 = x + x^3$

Click here to see video solution to problem #8(c)

Video errata: At the 41:18 mark, we should have that $\sum_{k=0}^{\infty} a_{2k+1}x^{2k+1} = a_1(x+x^3)$. At the 44:01 mark, we should have that $\sum_{k=0}^{\infty} a_{2k}x^{2k} = a_0\left(1 + 3x^2 + x^4 + \sum_{k=3}^{\infty} \frac{3(-1)^k}{(2k-1)(2k-3)}x^{2k}\right)$. This has been digitally added in the video.

(d) Find the first four terms in each of two solutions y_1 and y_2. Show that $W[y_1, y_2](0) \neq 0$.

Answer: $y_1 = 1 + 3x^2 + x^4 - \frac{1}{5}x^6 + ...$ $\quad y_2 = x + x^3$

$$W[y_1, y_2](0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$
$$= 1$$

Click here to see video solution to problem #8(d)

9. (Chapter 5) For the following equation, determine $\phi''(x_0)$ and $\phi'''(x_0)$, for the given point x_0 if $y = \phi(x)$ is a solution of the given initial-value problem.

$$y'' + x^2y' + (\sin x)y = 0; \quad y(0) = a_0, \quad y'(0) = a_1$$

Answer: We have that $\phi''(0) = 0$, $\phi'''(0) = -a_0$, and $\phi(x) = a_0(1 - \frac{x^3}{6} + ...) + a_1(x + ...)$

Click here to see video solution to problem #9

10. (Chapter 6) Find the solution of the initial value problem
   (a) $y'' + y = \delta(t - 2\pi)\cos t, \qquad y(0) = 0, \quad y'(0) = 1.$

   Answer: $y(t) = \sin(t) + u_{2\pi}(t)\sin(t)$

   Click here to see video solution to problem #10(a)

   (b) $y'' + 3y' + 2y = \begin{cases} 1, & 0 \le t < 10 \\ 0, & 10 \le t \end{cases}, \qquad y(0) = 0, \quad y'(0) = 0.$

   Answer: $y(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} - u_{10}(t)\left[\frac{1}{2} - e^{-(t-10)} + \frac{1}{2}e^{-2(t-10)}\right]$

   Click here to see video solution to problem #10(b)

11. (Chapter 6) Solve the initial value problem $y'' - 2y' - 3y = g(t)$, $y(0) = 1$, $y'(0) = -3$.

   Answer: $y(t) = -\frac{1}{2}e^{3t} + \frac{3}{2}e^{-t} + \frac{1}{4}\int_0^t g(t-\tau)(e^{3\tau} - e^{-\tau})d\tau$

   Click here to see video solution to problem #11

12. (Chapter 7) Find the general solution of the system and the fundamental matrix. Classify the type of the critical point (0,0), and determine whether it is stable or unstable. Sketch the phase portrait.

   (a) $\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}\mathbf{x}$

   Answer: The general solution is $\vec{x}(t) = C_1e^{2t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + C_2e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, the fundamental matrix is $\begin{pmatrix} e^{2t} & e^{4t} \\ 3e^{2t} & e^{4t} \end{pmatrix}$, and the critical point $(0, 0)$ is a nodal source and unstable. Refer to video for sketch of phase portrait.

   Click here to see video solution to problem #12(a)

(b) $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -5 & -1 \end{pmatrix} \mathbf{x}$

Answer: The general solution is $\vec{x}(t) = C_1 \begin{pmatrix} -\cos(2t) \\ \cos(2t) + 2\sin(2t) \end{pmatrix} + C_2 \begin{pmatrix} -\sin(2t) \\ \sin(2t) - 2\cos(2t) \end{pmatrix}$, the fundamental matrix is $\begin{pmatrix} -\cos(2t) & -\sin(2t) \\ \cos(2t) + 2\sin(2t) & \sin(2t) - 2\cos(2t) \end{pmatrix}$, and the critical point $(0,0)$ is a center and stable. Refer to video for sketch of phase portrait.

Click here to see video solution to problem #12(b)

(c) $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x}$

Answer: The general solution is $\vec{x}(t) = C_1 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, the fundamental matrix is $\begin{pmatrix} e^{-t} & 2e^{2t} \\ 2e^{-t} & e^{2t} \end{pmatrix}$, and the critical point $(0,0)$ is a saddle point and unstable. Refer to video for sketch of phase portrait.

Click here to see video solution to problem #12(c)

(d) $\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$

Answer: The general solution is $\vec{x}(t) = C_1 e^{t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{t} \left[t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]$, the fundamental matrix is $\begin{pmatrix} 2e^{t} & (2t+1)e^{t} \\ e^{t} & te^{t} \end{pmatrix}$, and the critical point $(0,0)$ is a improper node and unstable. Refer to video for sketch of phase portrait.

Click here to see video solution to problem #12(d)

(e) $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \mathbf{x}$

Answer: The general solution is

$$\vec{x}(t) = C_1 e^{-t} \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(t) \right] + C_2 e^{-t} \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} \sin(t) + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos(t) \right],$$

the fundamental matrix is $\begin{pmatrix} e^{-t}\cos(t) & e^{-t}\sin(t) \\ e^{-t}(2\cos(t) + \sin(t)) & e^{-t}(2\sin(t) - \cos(t)) \end{pmatrix}$, and the critical point $(0, 0)$ type is a spiral source and asymptotically stable. Refer to video for sketch of phase portrait.

Click here to see video solution to problem #12(e)