Week in Review Session ♯3 (Sections 2.4-2.6)

This document contains the answers and video solutions to the posed problems. Click the red box to watch the video solution. You can also watch all videos by viewing the Session 3 playlist. In the event that this weekly review handouts is updated, the Session video playlist will reflect the most updated problem set. Closed captions are available for all videos and the speed of the videos may be adjusted inside of "Settings" or the cog in the bottom right corner.

1. Determine an interval in which the solution of the following initial value problem is certain to exist.

$$(t^2-1)y' + (\sin t)y = \frac{\cot t}{t^2-4t+3}, \qquad y(2) = -1$$

Answer: $I = (1,3)$

Click here to see video solution to problem #1

2. State where in the ty-plane the hypothesis of the Existence and Uniqueness theorem are satisfied for the following differential equations.

(a) $y' = \dfrac{\ln(ty)}{1-(t^2+y^2)}$

Answer: Anywhere in the first or third quadrant, but not on the unit circle.

Click here to see video solution to problem #2(a)

(b) $y' = (t^2-y)^{1/3}$.

Answer: Any point off the parabola $y = t^2$.

Click here to see video solution to problem #2(b)

3. Solve the following initial value problems and determine how the interval in which the solution exists depends on the initial value y_0.

(a) $y' = \dfrac{-4}{t}y, \qquad y(2) = y_0$

Answer: If $y_0 = 0$ then $y = 0$ is an equilibrium, if $y_0 \neq 0$ then $y = \dfrac{16y_0}{t^4}$ and $I.V. = (0, \infty)$

Click here to see video solution to problem #3(a)

(b) $y' + y^3 = 0 \qquad y(t_0) = y_0$

Answer: If $y_0 = 0 \implies y = 0 \implies I = (-\infty, \infty)$. If $y_0 \neq 0$ then $I = (t_0 - \frac{1}{2{y_0}^2}, \infty)$

Click here to see video solution to problem #3(b)

4. Verify that both $y_1 = 1 - t$ and $y_2 = \frac{-t^2}{4}$ are solutions to the same initial value problem

$$y'(t) = \frac{-t + \sqrt{t^2 + 4y}}{2}, \qquad y(2) = -1.$$

Does it contradict the existence and uniqueness theorem?

Answer: There is no contradiction because the point $(2, 1)$ is not above the parabola $y = -\frac{t^2}{4}$. For verification that y_1 and y_2 are solutions to the same initial value problem, see video below.

Click here to see video solution to problem #4

Video errata: There is a mistake at the 11:00 mark when I wrote out f_y, we should have that $f_y = \frac{1}{2} \cdot \frac{1}{2}(t^2 + 4y)^{-\frac{1}{2}} \cdot (4) = \frac{1}{\sqrt{t^2 + 4y}}$. This does not change our answer as the condition on f_y is still $t^2 + 4y > 0$.

5. Given the differential equation

$$y' = y^3 - 4y$$

(a) Find the equilibrium solutions.
(b) Graph the phase line. Classify each equilibrium solution as either stable, unstable, semistable.
(c) Graph some solutions.
(d) If $y(t)$ is the solution of the equation satisfying the initial condition $y(0) = y_0$, where $-\infty < y_0 < \infty$, find the limit of $y(t)$ when t increases.

Answer: For parts (a-d), see videos below.

Click here to see video solution to problem #5(a) and #5(b)

Click here to see video solution to problem #5(c) and #5(d)

6. Given the differential equation

$$y'(t) = y^3 - 2y^2 + y$$

   (a) Find the equilibrium solutions.
   (b) Graph the phase line. Classify each equilibrium solution as either stable, unstable, or semistable.
   (c) Sketch the graph of some solutions.
   (d) If $y(t)$ is the solution of the equation satisfying the initial condition $y(0) = y_0$, where $-\infty < y_0 < \infty$, determine the behavior of $y(t)$ as t increases.
   (e) Do any solutions 'blow up in finite time,' namely, do they admit a vertical asymptote?

   Answer: For parts (a-e), see videos below.

   Click here to see video solution to problem #6(a) and #6(b)

   Click here to see video solution to problem #6(c),#6(d), and #6(e)

7. Determine if the equations are exact and solve the ones that are:
   (a) $(2x + 5y)dx + (5x - 6y)dy = 0.$

   Answer: Equation is exact, $x^2 + 5xy - 3y^2 = C.$

   Click here to see video solution to problem #7(a)

   (b) $1 + \dfrac{y}{x} - \dfrac{1}{x}y' = 0$

   Answer: Equation is not exact.

   Click here to see video solution to problem #7(b)

   (c) $(\sin(2t) + 2y)dy + (2y\cos(2t) - 6t^2)dt = 0$

   Answer: Equation is exact, $y\sin(2t) + y^2 - 2t^3 = C$

   Click here to see video solution to problem #7(c)

8. Show that the following equations are not exact. However, if you multiply by the given integrating factor, the resulting equation is exact.

   (a) $(x^2 + y^2 - x)dx - ydy = 0 \qquad \mu(x, y) = \dfrac{1}{x^2 + y^2}$

   Answer: For verification, see video below.

   Click here to see video solution to problem #8(a)

   (b) $3(y + 1)dx - 2xdy = 0, \qquad \mu(x, y) = \dfrac{y + 1}{x^4}$

   Answer: For verification, see video below.

   Click here to see video solution to problem #8(b)

9. Find an integrating factor for the equation

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

and then solve the equation.

   Answer: $\mu = x, \quad x^3y + \frac{1}{2}x^2y^2 = C$

   Click here to see video solution to problem #9

10. Solve the IVP $(3x^2 + 2xy^2)dx + 2x^2ydy = 0,\ y(2) = -3$.

   Answer: $y = -\sqrt{\dfrac{44 - x^3}{x^2}}$

   Click here to see video solution to problem #10

11. Solve $(x^4 \ln x - 2xy^3)dx + 3x^2y^2dy = 0$.

   Answer: $x\ln(x) - x + x^{-2}y^3 = C$

   Click here to see video solution to problem #11

12. Solve $y' = \dfrac{(1+x)e^x}{xe^x - ye^y}$.

   Answer: $\dfrac{y^2}{2} + xe^x e^{-y} = C$

   Click here to see video solution to problem #12

13. Solve

$$ydx = (y^2 + x^2 + x)dy,$$

   if we know there is an integrating factor of the form $\mu = \phi(x^2 + y^2)$.

   Answer: $\arctan\left(\dfrac{x}{y}\right) - y = C$

   Click here to see video solution to problem #13