Week in Review Session ♯6 (Sections 3.7-3.8) & Exam 1 Review

This document contains the answers and video solutions to the posed problems. Click the red box to watch the video solution. You can also watch all videos by viewing the Session 6 playlist. In the event that this weekly review handout is updated, the Session video playlist will reflect the most updated problem set. Closed captions are available for all videos and the speed of the videos may be adjusted inside of "Settings" or the cog in the bottom right corner.

1. A mass weighing 3 lb stretches a spring 3 in. If the mass is pushed upward, contracting the spring a distance of 1 in, then set in motion with a downward velocity of 2 ft/s, and if there is no damping, find the position u of the mass at any time t. Determine the frequency, period, amplitude and phase angle of the motion.

   Answer: $u(t) = -\frac{1}{12}\cos(8\sqrt{2}t) + \frac{1}{4\sqrt{2}}\sin(8\sqrt{2}t)$, the frequency w_0 is $8\sqrt{2}\,\frac{\text{rad}}{\text{s}}$, the period T_0 is $\frac{\pi}{4\sqrt{2}}$ seconds, the amplitude R is $\frac{\sqrt{22}}{24}$ ft , and the phase angle of motion δ is $\arctan(-\frac{3}{\sqrt{2}}) + \pi$ radians.

   Click here to see video solution to problem #1

2. A spring is stretched 10 cm by a force of 3 N. A mass of 2 kg is hung from the spring and is also attached to a viscous damper that exerts a force of 3 N when the velocity of the mass 5 m/s. If the mass is pulled down 5 cm below its equilibrium position and given an initial velocity of 10 cm/s, determine its position u at any time. Find the quasifrequency of the motion.

   Answer: $u = e^{-\frac{3}{20}t}[0.05\cos(\mu t) + \frac{2.15}{\sqrt{5991}}\sin(\mu t)]$ where the quasifrequency $\mu = \frac{\sqrt{5991}}{20}$.

   Click here to see video solution to problem #2

3. A spring is stretched 6 in by a mass that weighs 8 lb. The mass is attached to a dashpot mechanism that has a damping constant of 0.25 lb·s/ft and is acted by an external force of $4\cos 2t$ lb.

   (a) Find the steady-state response of this system.

   Answer: $u_p = u_{st} = \frac{120}{450.5}\cos(2t) + \frac{4}{450.5}\sin(2t)$

   Click here to see video solution to problem #3(a)

(b) If the given mass is replaced by a mass m, determine the value of m for which the amplitude of the steady-state response is maximum.

Answer: $m = 4$ slugs

Click here to see video solution to problem #3(b)

(c) If the mass is the same as in the problem, determine the value ω of the frequency of the external force $4\cos\omega t$ lb at which "*practical resonance*" occurs, i.e., the amplitude of the steady-state response is maximized.

Answer: $w_{res} = \sqrt{63.5}\ \dfrac{\text{rad}}{\text{sec}}$

Click here to see video solution to problem #3(c)

4. A mass weighing 4 lb stretches a spring 1.5 in. The mass is given a positive displacement 2 in from its equilibrium position and released with no initial velocity. Assuming that there is no damping and the mass is acted on by an external force of $2\cos 3t$ lb,

(a) Formulate the initial value problem describing the motion of mass

Answer:

$$\begin{cases} \dfrac{1}{8}x'' + 32x = 2\cos(3t) \\ \\ x(0) = \dfrac{1}{6}\text{ ft},\ \ x'(0) = 0 \end{cases}$$

Click here to see video solution to problem #4(a)

(b) Solve the initial value problem.

Answer: $x = \dfrac{151}{247(6)}\cos(16t) + \dfrac{16}{247}\cos(3t)$

Click here to see video solution to problem #4(b)

Video errata: In part (b), 12:20 into the video, when solving for C_1, I initially wrote $C_1 = \dfrac{247-96}{247}$. This was edited in the video to display $C_1 = \dfrac{247-96}{247(6)}$ and was edited in any place where C_1 appeared. I also made note of this change at the beginning of part (c).

(c) If the given external force is replaced by a force $4\cos\omega t$ of frequency ω, find the value of ω for which resonance occurs.

Answer: $w = 16\ \frac{\text{rad}}{\text{s}}$

Click here to see video solution to problem #4(c)

5. A 3 kg object is attached to a spring and will stretch the spring 392 mm by itself. There is no damping in the system and a forcing function of the form $F(t) = 10\cos(\omega t)$ is attached to the object and the system will experience resonance. If the object is initially displaced 20 cm downward from its equilibrium position and given a velocity of 10 cm/sec upward find the displacement at any time t.

Answer: $x = 0.2\cos(5t) - 0.02\sin(5t) + \frac{1}{3}t\sin(5t)$

Click here to see video solution to problem #5

Review for Exam 1

6. For the initial value problem $(t^2-4)y' + 2ty = 3t^2, \quad y(1) = -3$

(a) Determine an interval in which the solution to the initial value problem is certain to exist.

Answer: $I = (-2, 2)$

Click here to see video solution to problem #6(a)

(b) Solve the initial value problem.

Answer: $y = \dfrac{t^2 - 2t + 4}{t-2}, \quad I.V. = (-\infty, 2)$

Click here to see video solution to problem #6(b)

7. A large tank initially contains 10 L of fresh water. A brine containing 20 g/L of salt flows into the tank at a rate of 3 L/min. The solution inside the tank is kept well stirred and flows out of the tank at a rate of 2 L/min. Determine the concentration of salt in the tank as a function of time.

Answer: $C(t) = 20 - 20,000(t+10)^{-3}$

Click here to see video solution to problem #7

8. Given the differential equation

$$\frac{dy}{dt} = 7y - y^2 - 10$$

(a) Find the equilibrium solutions.
(b) Sketch the phase line and determine whether the equilibrium solutions are stable, unstable, or semistable.
(c) Sketch the graph of some solutions.
(d) Determine the behavior of $y(t)$ as t increases for all possible values of $y(0) = y_0$.
(e) Do any solutions admit a vertical asymptote?
(f) Solve the equation.

Answer: For parts (a-f), see videos below.

Click here to see video solution to problem #8(a-e)

Click here to see video solution to problem #8(f)

9. Find an integrating factor for the equation

$$(y^2 + xy) + (x^2 + 3xy)y' = 0$$

and then solve the equation.

Answer: $\frac{1}{2}x^2y^2 + xy^3 = C$

Click here to see video solution to problem #9