Week in Review Session ♯8 (Sections 6.4-6.6)

This document contains the answers and video solutions to the posed problems. Click the red box to watch the video solution. You can also watch all videos by viewing the Session 8 playlist. In the event that this weekly review handout is updated, the Session video playlist will reflect the most updated problem set. Closed captions are available for all videos and the speed of the videos may be adjusted inside of "Settings" or the cog in the bottom right corner.

1. Solve the following initial value problem using the Laplace transform:

$$y'' + 2y' + 5y = \sin(t) + u_\pi(t)\cos(t - \pi), \qquad y(0) = 0, \quad y'(0) = 0.$$

   Answer:
   $y(t) = h(t) + u_\pi(t)p(t - \pi)$, where $h(t) = -\frac{1}{10}\cos(t) + \frac{2}{10}\sin(t) + e^{-t}[\frac{1}{10}\cos(2t) - \frac{1}{20}\sin(2t)]$
   and $p(t - \pi) = -\frac{2}{10}\cos(t) - \frac{1}{10}\sin(t) + e^{-(t-\pi)}[-\frac{2}{10}\cos(2t) - \frac{3}{20}\sin(2t)]$

   Click here to see video solution to problem #1

2. Find the solution of the initial value problem
   $y'' - y = 4\delta(t - 2) + t^2, \qquad y(0) = 0, \quad y'(0) = 2.$

   Answer: $x(t) = -2 - t^2 + 2\cosh(t) + 2\sinh(t) + 4u_2(t)\sinh(t - 2)$

   Click here to see video solution to problem #2

3. A 2-kg mass is attached to a spring with the Hooke's constant of 3 N/m, and it is subject to moving in a medium with a damping constant of 5 N-s/m. The mass is initially displaced +0.5 m from its equilibrium position and is released. Then after 2 seconds, a hammer hits the mass in such a way that its velocity suddenly changes by 5m/s in the positive direction (i.e., $\Delta v = +5$ m/s).

   (a) Find the impulse of the hammer and its units.

   Answer: $I = 10\ N \cdot s$

   Click here to see video solution to problem #3(a)

   (b) Write the hammer force using an appropriate Dirac delta function. Note that your force must result in the same impulse value you found in part (a).

   Answer: $F(t) = 10\delta(t - 2)$

   Click here to see video solution to problem #3(b)

(c) Set up an IVP and find the position $x(t)$ of the mass for $t \geq 0$.

Answer: $x(t) = -e^{-\frac{3}{2}t} + 1.5e^{-t} + u_2(t)\left(10e^{-(t-2)} - 10e^{-\frac{3}{2}(t-2)}\right)$

Click here to see video solution to problem #3(c)

4. Find the following convolutions using the definition only.

(a) $e^t * e^{3t}$

Answer: $\frac{1}{2}e^{3t} - \frac{1}{2}e^t$

Click here to see video solution to problem #4(a)

(b) $t * t^n$, where $n = 0, 1, 2, \ldots$.

Answer: $\frac{t^{n+2}}{(n+1)(n+2)}$

Click here to see video solution to problem #4(b)

5. Using the Laplace transform (instead of the definition), compute the following convolutions.

(a) $u_a(t) * u_b(t)$

Answer: $u_{a+b}(t)(t - (a+b))$

Click here to see video solution to problem #5(a)

(b) $t^n * t^m,\ n, m = 0, 1, 2, \ldots$

Answer: $\frac{n!m!}{(n+m+1)!}t^{n+m+1}$

Click here to see video solution to problem #5(b)

6. In each of the following cases find a function (or a generalized function) $g(t)$ that satisfies the equality for $t \geq 0$.

(a) $t * g(t) = t^4$

Answer: $g(t) = 12t^2$

Click here to see video solution to problem #6(a)

(b) $1 * 1 * g(t) = t^2$

Answer: $g(t) = 2$

Click here to see video solution to problem #6(b)

(c) $1 * g(t) = 1$

Answer: $g(t) = \delta(t)$

Click here to see video solution to problem #6(c)

7. Write the inverse Laplace transform of the function $F(s) = \dfrac{s}{(s+1)^2(s+4)^3}$ in terms of a convolution integral.

   Answer: $f(t) = \displaystyle\int_0^t e^{-\tau}(\tau - 1) \cdot \frac{1}{2}(t-\tau)^2 \cdot e^{-4(t-\tau)} d\tau$

   Click here to see video solution to problem #7

8. Solve the initial value problem $y'' - 2y' - 3y = g(t),\ y(0) = 1,\ y'(0) = -3.$

   Answer: $y(t) = -\dfrac{1}{2}e^{3t} + \dfrac{3}{2}e^{-t} + \dfrac{1}{4}\displaystyle\int_0^t g(t-\tau)(e^{3\tau} - e^{-\tau}) d\tau$

   Click here to see video solution to problem #8