Note ♯12: Sections 7.1-7.3

Problem 1. Transform the given equation into a system of first-order equations.

(a) $tu'' + u' + 4u = 2\cos(3t), \quad u(0) = 1, \quad u'(0) = -2$

(b) $u^{(4)} - u'' + u = 0$

Problem 2. Transform the given system into a single equation of second order and find x_1 and x_2.

(a) $x_1' = -2x_1 + x_2, \quad x_2' = x_1 - 2x_2$

(b) $x_1' = 3x_1 - 2x_2, \quad x_2' = 2x_1 - 2x_2, \quad x_1(0) = 3, \quad x_2(0) = \frac{1}{2}$

Problem 3. If $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & -2 \\ -1 & 5 \end{pmatrix}$, find

(a) $\mathbf{A}^T$ and $\mathbf{B}^T$

(b) $\mathbf{A}^T + \mathbf{B}^T$ and $(\mathbf{A} + \mathbf{B})^T$

(c) $2\mathbf{A} + \mathbf{B}$

(d) $\mathbf{A} - 4\mathbf{B}$

(e) $\mathbf{AB}$ and $\mathbf{BA}$

Problem 4. If $\mathbf{A} = \begin{pmatrix} 1+i & -1+2i \\ 3+2i & 2-i \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} i & 3 \\ 2 & -2i \end{pmatrix}$, find

(a) $\mathbf{A} - 2\mathbf{B}$

(b) $3\mathbf{A} + \mathbf{B}$

(c) $\mathbf{AB}$

(d) $\mathbf{BA}$

Problem 5. If $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4 & -2 \\ -1 & 0 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$, $\alpha = 2$, and $\beta = -1$, check these useful identities:

(a) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

(b) $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$

(c) $\alpha(\mathbf{A} + \mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B}$

(d) $(\alpha + \beta)\mathbf{A} = \alpha\mathbf{A} + \beta\mathbf{A}$

(e) $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$

(f) $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$

Problem 6. Check if the matrix is singular. If the matrix is nonsingular, find its inverse.

(a) $\begin{pmatrix} 1 & 4 \\ -2 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 3 & -1 \\ 6 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

Problem 7. If $\mathbf{A}(t) = \begin{pmatrix} e^t & 2e^{-t} \\ 2e^t & e^{-t} \end{pmatrix}$ and $\mathbf{B}(t) = \begin{pmatrix} 2e^t & e^{-t} \\ -e^t & 2e^{-t} \end{pmatrix}$, find

(a) $\mathbf{A} + 3\mathbf{B}$

(b) $\mathbf{AB}$

(c) $\frac{d\mathbf{A}}{dt}$

(d) $\int_0^1 \mathbf{A}(t)dt$

Problem 8. Verify that the given vector/matrix satisfies the given differential equation.

(a) $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$, $\mathbf{x} = \begin{pmatrix} (1+2t)e^t \\ 2te^t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^t$

(b) $\Psi' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \Psi, \quad \Psi(t) = \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}$

Problem 9. Either solve the given system of equations, or else show that there is no solution.

(a) $$\begin{aligned} x_1 + 2x_2 &= 1 \\ 3x_1 - 2x_2 &= 2 \end{aligned}$$

(b) $$\begin{aligned} x_1 + 2x_2 &= 2 \\ 2x_1 + 4x_2 &= 1 \end{aligned}$$

(c) $$\begin{aligned} x_1 + 2x_2 &= 0 \\ -x_1 - 2x_2 &= 0 \end{aligned}$$

Problem 10. Determine whether the members of the given set of vectors are linearly independent for $-\infty < t < \infty$. If they are linearly dependent, find the linear relation among them.

(a) $\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

(b) $\mathbf{x}^{(1)}(t) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$

Problem 11. Find all eigenvalues and eigenvectors of the given matrix.

(a) $\begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$