Note ♯14: Final Review

Problem 1. Use the integrating factor method for solving the problem

$$ty' + (t+1)y = t, \quad y(\ln 2) = 1, \quad t > 0.$$

Problem 2. Solve the initial value problem with separation of variables

$$y' = \left(3x^2 - e^x\right)/(2y - 5), \quad y(0) = 1.$$

Problem 3. Determine the critical (equilibrium) points, and classify each one as asymptotically stable/semistable or unstable.

$$dy/dt = y(y-1)^2(y-2), \quad y_0 \geq 0.$$

Problem 4. Determine whether the equation is exact. If it is exact, find the solution.

$$(y/x + 6x) + (\ln x - 2)y' = 0, \quad x > 0$$

Problem 5. Find the general solutions:

(1) $y'' - 5y' + 6y = 0$
(2) $y'' - 6y' + 9y = 0$
(3) $y'' - 4y' + 5y = 0$

Problem 6. Suppose y_1 and y_2 are homogeneous solutions of a second order differential equation for $x > 0$. Do they constitute a fundamental set of solutions?

$y_1(x) = e^{-x}, \quad y_2(x) = xe^{-x}$

Problem 7. Find the form of a particular solution for the method of undetermined coefficients. Do not solve for the unknown constants.

(1) $y'' - 5y' + 6y = e^{2t}(3t + 4)\sin t$
(2) $y'' + 2y' + y = e^{-t}\,(t^2 + 1)$

Problem 8. y_1 and y_2 form a fundamental set of solutions of the corresponding homogeneous equation. Find the general solution of the given nonhomogeneous equation using variation of parameters.

$$t^2y'' - 2y = 3t^2 - 1, t > 0; \quad y_1(t) = t^2, y_2(t) = t^{-1}$$

Problem 9. Find the Laplace transform using the definition of a Laplace transform.

$$f(t) = \begin{cases} \cos(t)\delta(t - \pi/4), & 0 \leq t < 10 \\ t, & 10 \leq t < \infty \end{cases}$$

Problem 10. Rewrite the function using unit step functions.

$$f(t) = \begin{cases} t-2, & 0 \le t < 3 \\ 5, & 3 \le t < 7 \\ 0, & 7 \le t < \infty \end{cases}$$

Problem 11. Find the solution of the given initial value problem.

$$y'' + 4y = u_{2\pi}(t) - u_{4\pi}(t); \quad y(0) = 0, \quad y'(0) = 0$$

Problem 12. Find the inverse Laplace transform of the given function by using the convolution theorem. **Do not evaluate the integral.**

$$F(s) = \frac{s}{(s-2)^2\,(s^2+4)}$$

Problem 13. a. What is the minimum radius of convergence of the solutions?

b. Find power series solutions of the given differential equation about $x_0 = 0$. Clearly indicate two solutions y_1 and y_2 by finding the first three nonzero terms in each solution.

$$(1 - x)y'' + xy' - y = 0, \quad y(0) = -3, \quad y'(0) = 2$$

Problem 14. Change the following differential equation into a system of first order differential equations.

$$y''' - 2ty + 3 = 0$$

Problem 15. Solve the given initial value problem. Describe the behavior of the solution as $t \to \infty$.

$$\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Problem 16. Find the general solution. What kind of point is the origin?(saddle/node/spiral point)

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \mathbf{x}$$

Problem 17. Find the general solution of the system of equations using linear algebra. What kind of point is the origin?(saddle/node/spiral point)

$$x_1' = 4x_1 - 2x_2$$
$$x_2' = 8x_1 - 4x_2$$