Problem 1. Identify the differential equation that corresponds to the given direction field. Based on the direction field, determine the behavior of \( y \) as \( t \to \infty \). If this behavior depends on the initial value of \( y \) at \( t = 0 \), describe the dependency.

- a. \( y' = y - 2 \)
- b. \( y' = y - 3 \)
- c. \( y' = 2 + y \)
- d. \( y' = y - 2 \)
- e. \( y' = y(y - 3) \)
- f. \( y' = 1 + 2y \)
- g. \( y' = y(y - 3) \)
- h. \( y' = y(y - 3) \)

- i. \( y' = 1 - 2y \), \( y' = 2 - 2y \)

\( y > 2 \): \( y' < 0 \)

\( y = 2 \): \( y' = 0 \)

\( y < 2 \): \( y' > 0 \)

9. \( y > 2 \) as \( t \to \infty \)

- \( y > 2 \) if \( y(0) > 2 \)
- \( y \to -\infty \) if \( y(0) < 2 \)

- \( y < 0 \): \( y' < 0 \)
- \( y = 0 \): \( y' > 0 \)
- \( y > 0 \): \( y' < 0 \)

- \( 0 < y < 2 \): \( y' > 0 \)
- \( y < 2 \): \( y' < 0 \)
- \( y = 2 \): \( y' = 0 \)

- \( y > 3 \): \( y \to y = 3 \)
- \( y > 3 \): \( y \to \infty \)

- \( 0 < y < 3 \): \( y \to y = 3 \)
- \( 0 < y < 3 \): \( y \to 0 \)

- \( y < 0 \): \( y \to -\infty \)
- \( y < 0 \): \( y \to 0 \)
Problem 2. For small, slowly falling objects, the assumption that the drag force is proportional to the velocity is a good one. For larger, more rapidly falling objects, it is more accurate to assume that the drag force is proportional to the square of the velocity.

\[ F = \gamma \cdot v^2 \]

a. Write a differential equation for the velocity of a falling object of mass \( m \) if the magnitude of the drag force is proportional to the square of the velocity and its direction is opposite to that of the velocity.

\[ m \cdot \frac{dv}{dt} = mg - \gamma \cdot v^2 \]

b. Determine the limiting velocity after a long time.

c. If \( m = 10 \, \text{kg} \), find the drag coefficient so that the limiting velocity is \( 40 \, \text{m/s} \).

\[ 0 = mg - \gamma \cdot v^2 \quad \Leftrightarrow \quad v = \sqrt{\frac{mg}{\gamma}} \]

\[ 0 = 10 \cdot 9.8 - \gamma \cdot (49)^2 \]

\[ \gamma = \frac{9.8}{(49)^2} = \frac{2}{49} \]
**Problem 3.** According to Newton's law of cooling, the temperature \( u(t) \) of an object satisfies the differential equation

\[
\frac{du}{dt} = -k(u-T),
\]

where \( T \) is the constant ambient temperature and \( k \) is a positive constant. Suppose that the initial temperature of the object is \( u(0) = u_0 \).

a. Find the temperature of the object at any time.

\[
(0) \quad \frac{du}{u-T} = -k \, dt \quad \Rightarrow \quad \ln |u-T| = -kt + C_0
\]

\[
\Leftrightarrow |u-T| = e^{-kt+C_0} = e^{-kt} \cdot e^{C_0} = c_1 e^{-kt}
\]

\[
\Leftrightarrow u-T = \pm c_1 e^{-kt}
\]

\[
\Leftrightarrow u-T = C_2 e^{-kt}
\]

\[
\text{Only one of } +C_1 \text{ or } -C_1 \text{ will occur.}
\]

\[
\Rightarrow u = T + C_2 e^{-kt}
\]

b. Let \( \tau \) be the time at which the initial temperature difference \( u_0 - T \) has been reduced by half. Find the relation between \( k \) and \( \tau \).

\[
\Rightarrow u_0 = T + C_2 e^0 = T + C_2
\]

\[
\Rightarrow C_2 = u_0 - T
\]

\[
\Rightarrow u = T + (u_0 - T) e^{-kt}
\]

\[
(\text{b}) \quad u(T) - T = \frac{1}{2} (u_0 - T)
\]

\[
\Rightarrow u(T) = T + \frac{1}{2} (u_0 - T) = \frac{1}{2} T + \frac{1}{2} u_0
\]

\[
\Rightarrow C_2 = \frac{1}{2} u_0
\]

\[
\Rightarrow e^{-k\tau} = \frac{1}{2} \left( u_0 - T \right)
\]

\[
\Leftrightarrow -k\tau = \ln(\frac{1}{2})
\]

\[
\tau = \frac{-\ln(\frac{1}{2})}{k}
\]
Problem 4. Determine the order of the given differential equation; also state whether the equation is linear or nonlinear.

1. \( \frac{d^2 y}{dt^2} + \frac{d y}{dt} + 2y - \sin t. \)
   \( \begin{align*}
   \text{order: } & 2, \quad \text{linear} \\
   \text{order: } & 2, \quad \text{nonlinear}
   \end{align*} \)

2. \( (1 + y^2) \frac{d^2 y}{dt^2} - \frac{d y}{dt} + y = e^t. \)
   \( \begin{align*}
   \text{order: } & 2, \quad \text{nonlinear}
   \end{align*} \)

3. \( \frac{d^4 y}{dt^4} + \frac{d^3 y}{dt^3} + \frac{d^2 y}{dt^2} + \frac{dy}{dt} + y = 1. \)
   \( \begin{align*}
   \text{order: } & 4, \quad \text{linear}
   \end{align*} \)

4. \( \frac{d^2 y}{dt^2} + \sin(t + y) = \sin t. \)
   \( \begin{align*}
   \text{order: } & 2, \quad \text{nonlinear}
   \end{align*} \)

5. \( 2y_{xxxx} + 4y_{xxx} + 6y_{xx} = 0. \)
   \( \begin{align*}
   \text{order: } & 4, \quad \text{linear}
   \end{align*} \)

6. \( \frac{d^2 u}{dx^2} + u = 2f_{xx} \)
   \( \begin{align*}
   \text{order: } & 2, \quad \text{nonlinear}
   \end{align*} \)

- \( y(t) \)
  \( y^2 = 1 \rightarrow \text{nonlinear} \)
- \( u(y) \)
  \( y^2 u = 1 \rightarrow \text{linear} \)
- \( \sin(\theta) y(\theta) + y(\theta) = 0 \)
  \( \sin(\theta) y(\theta) + y(\theta) = 0 \)
**Problem 5.** Verify that each given function is a solution of the differential equation.

\[ t^2y'' + 5ty' - 4y = 0, \quad t > 0; \quad y_1(t) = t^{-2}, \quad y_2(t) = t^{-2} \ln t. \]

\[ y_2 = t^{-2} \ln t \]
\[ y_2' = -2t^{-3} \ln t + t^{-2} \cdot t^{-1} \]
\[ = t^{-3} (\ln t - 1) \]
\[ y_2'' = -3 \cdot t^{-4} (\ln t + 1) + t^{-3} (\ln t - 1) \]
\[ = t^{-4} (6 \ln t - 1) \]

\[ t^2 \left( t^{-4} (6 \ln t - 1) \right) + 5t \left( t^{-3} (\ln t - 1) \right) + 4t^{-2} \ln t \]
\[ = t^2 (6 \ln t - 1) + 5t (\ln t - 1) + 4t^{-2} \ln t \]
\[ = 0 \]
\[ \Rightarrow y_2 \text{ is a solution} \]
Problem 6. Determine the values of $r$ for which the given differential equation has solutions of the form $y = e^{rt}$.

\[ y'' + 3y' + 2y = 0. \]

Plugging $y = e^{rt}$,

\[ (r^2e^{rt}) + 3(re^{rt}) + 2(e^{rt}) = 0 \]

\[ \iff (r^2 + 3r + 2)e^{rt} = 0 \]

\[ \iff r^2 + 3r + 2 = 0 \]

\[ \iff (r+2)(r+1) = 0 \iff r = -2, -1 \iff y = e^{-2t}, y = e^{-t} \]

Problem 7. Determine the values of $r$ for which the given differential equation has solutions of the form $y = t^r$ for $t > 0$.

\[ t^2y'' + 4ty' + 2y = 0. \]

\[
\begin{align*}
\begin{cases}
y = t^r \\
y' = rt^{r-1} \\
y'' = r(r-1)t^{r-2}
\end{cases}
\end{align*}
\]

\[ \implies t^2(r(r-1)t^{r-2}) + 4t(rt^{r-1}) + 2t^r = 0 \]

\[ \iff r(r-1)t^r + 4rt^{r+1} + 2t^r = 0 \]

\[ \iff (r(r-1)+4r+2)t^r = 0 \]

\[ \iff (r^2+3r+2) = 0 \]

\[ \iff (r+2)(r+1) = 0 \]

\[ r = -2, -1 \]

\[ t > 0 \]

\[ \text{So, divide by } t^r \]